

UNIT 7

When sport balls are kicked, thrown, or hit into the air, the flight paths are parabolas that can be described by quadratic functions like $y = -16x^2 + 40x + 5$. Quadratic functions also provide models for the shape of suspension bridge cables, television dish antennas, and the graphs of revenue and profit functions in business.

The understanding and skill you need to solve problems involving quadratic functions will develop from your work on problems in three lessons of this unit.

QUADRATIC FUNCTIONS



Lessons

1 Quadratic Patterns

Explore typical quadratic relations and expressions to discover and explain connections among problem conditions, data tables, graphs, and function rules for quadratic patterns of change.

2 Equivalent Quadratic Expressions

Use algebraic properties of number systems to write quadratic expressions in convenient equivalent forms.

3 Solving Quadratic Equations

Solve quadratic equations by algebraic methods, including factoring and the quadratic formula.

LESSON 1



Quadratic Patterns

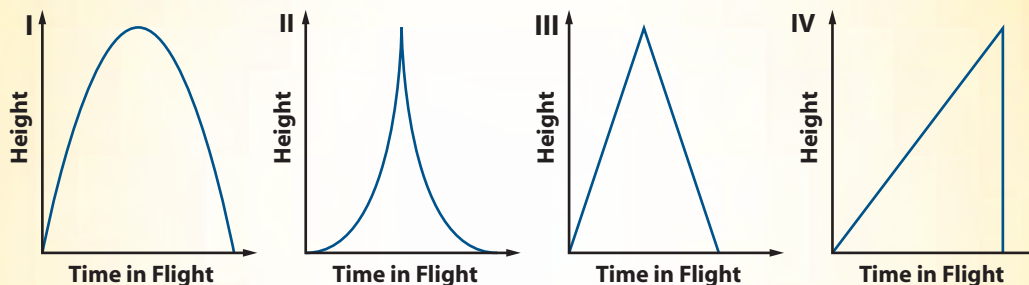
The town of Rehoboth Beach, Delaware, is a popular summer vacation spot along the Atlantic Ocean coast. When Labor Day passes, most beach people leave until the following summer. However, many return in the Fall for the annual *Punkin' Chunkin'* festival. The main attraction of this weekend is a “World Championship” contest to see which team of amateur engineers devised the best machine for launching pumpkins a long distance.



Think About This Situation

The current distance record for Punkin' Chunkin' is over 4,000 feet. Such a flight would take the pumpkin very high in the air, as well.

- a Which of these graphs is most likely to fit the pattern relating pumpkin height to time in flight? Explain your choice.



- b What pattern would you expect to find in data tables relating pumpkin height to elapsed time?

In work on investigations of this lesson, you will explore several strategies for recognizing, modeling, and analyzing patterns like those involved in the motion of a flying pumpkin.

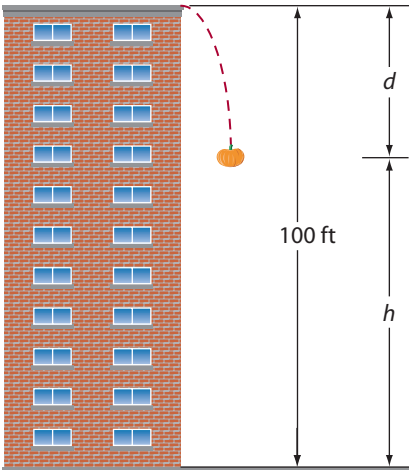
Investigation 1 Pumpkins in Flight

It turns out that the height of a flying pumpkin can be modeled well by a *quadratic* function of elapsed time. You can develop rules for such functions by reasoning from basic principles of science. Then you can use a variety of strategies to answer questions about the relationships. As you work on the problems of this investigation, look for answers to these questions:

What patterns of change appear in tables and graphs of (time, height) values for flying pumpkins and other projectiles?

What functions model those patterns of change?

Punkin' Droppin' At Old Dominion University in Norfolk, Virginia, physics students have their own flying pumpkin contest. Each year they see who can drop pumpkins on a target from 10 stories up in a tall building while listening to music by the group Smashing Pumpkins.



By timing the flight of the falling pumpkins, the students can test scientific discoveries made by Galileo Galilei, nearly 400 years ago. Galileo used clever experiments to discover that gravity exerts force on any free-falling object so that d , the distance fallen, will be related to time t by the function

$$d = 16t^2 \text{ (time in seconds and distance in feet).}$$

For example, suppose that the students dropped a pumpkin from a point that is 100 feet above the ground. At a time 0.7 seconds after being dropped, the pumpkin will have fallen $16(0.7)^2 \approx 7.84$ feet, leaving it $100 - 7.84 = 92.16$ feet above the ground.

This model ignores the resisting effects of the air as the pumpkin falls. But, for fairly compact and heavy objects, the function $d = 16t^2$ describes motion of falling bodies quite well.

- 1

Create a table like the one below to show estimates for the pumpkin’s distance fallen and height above ground in feet at various times between 0 and 3 seconds.

Time t	Distance Fallen d	Height Above Ground h
0	0	100
0.5	4	$100 - 4 = 96$
1		
1.5		
2		
2.5		
3		

- 2 Use data relating height and time to answer the following questions about flight of a pumpkin dropped from a position 100 feet above the ground.
- a. What function rule shows how the pumpkin's height h is related to time t ?
 - b. What equation can be solved to find the time when the pumpkin is 10 feet from the ground? What is your best estimate for the solution of that equation?
 - c. What equation can be solved to find the time when the pumpkin hits the ground? What is your best estimate for the solution of that equation?
 - d. How would your answers to Parts a, b, and c change if the pumpkin were to be dropped from a spot 75 feet above the ground?

High Punkin' Chunkin' Compressed-air cannons, medieval catapults, and whirling slings are used for the punkin' chunkin' competitions.



Imagine pointing a punkin' chunkin' cannon straight upward. The pumpkin height at any time t will depend on its speed and height when it leaves the cannon.

- 3 Suppose a pumpkin is fired straight upward from the barrel of a compressed-air cannon at a point 20 feet above the ground, at a speed of 90 feet per second (about 60 miles per hour).
- a. If there were no gravitational force pulling the pumpkin back toward the ground, how would the pumpkin's height above the ground change as time passes?
 - b. What function rule would relate height above the ground h in feet to time in the air t in seconds?

- c. How would you change the function rule in Part b if the punkin' chunker used a stronger cannon that fired the pumpkin straight up into the air with a velocity of 120 feet per second?
- d. How would you change the function rule in Part b if the end of the cannon barrel was only 15 feet above the ground, instead of 20 feet?

4 Now think about how the flight of a launched pumpkin results from the combination of three factors:

- initial height of the pumpkin's release,
 - initial upward velocity produced by the pumpkin-launching device, and
 - gravity pulling the pumpkin down toward the ground.
- a. Suppose a compressed-air cannon fires a pumpkin straight up into the air from a height of 20 feet and provides an initial upward velocity of 90 feet per second. What function rule would combine these conditions and the effect of gravity to give a relation between the pumpkin's height h in feet and its flight time t in seconds?
 - b. How would you change your function rule in Part a if the pumpkin is launched at a height of 15 feet with an initial upward velocity of 120 feet per second?

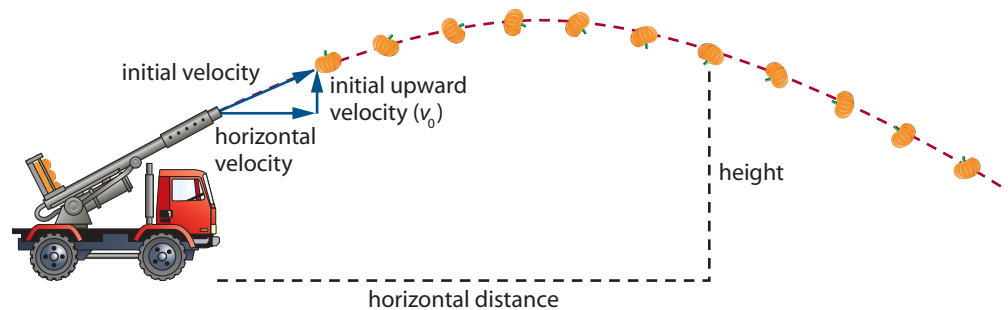
5 By now you may have recognized that the height of a pumpkin shot straight up into the air at any time in its flight will be given by a function that can be expressed with a rule in the general form

$$h = h_0 + v_0t - 16t^2.$$

In those functions, h is measured in feet and t in seconds.

- a. What does the value of h_0 represent? What units are used to measure h_0 ?
- b. What does the value of v_0 represent? What units are used to measure v_0 ?

When a pumpkin is not launched straight up into the air, we can break its velocity into a vertical component and a horizontal component. The vertical component, the *upward velocity*, can be used to find a function that predicts change over time in the pumpkin's height. The horizontal component can be used to find a function that predicts change over time in the horizontal distance traveled.



6 The pumpkin's height in feet t seconds after it is launched will still be given by $h = h_0 + v_0t - 16t^2$. It is fairly easy to measure the initial height (h_0) from which the pumpkin is launched, but it is not so easy to measure the initial upward velocity (v_0).

- a. Suppose that a pumpkin leaves a cannon at a point 24 feet above the ground when $t = 0$. What does that fact tell about the rule giving height h as a function of time in flight t ?
- b. Suppose you were able to use a stopwatch to discover that the pumpkin shot described in Part a returned to the ground after 6 seconds. Use that information to find the value of v_0 .

7 Suppose that you were able to use a ranging tool that records the height of a flying pumpkin every half-second from the time it left a cannon. A sample of the data for one pumpkin launch appears in the following table.

Time (in seconds)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Height (in feet)	15	40	60	70	70	65	50	30	0

- a. Plot the data on a graph and experiment with several values of v_0 and h_0 in search of a function that models the data pattern well.
- b. Use a calculator or computer tool that offers quadratic curve-fitting to find a quadratic model for the sample data pattern. Compare that automatic curve-fit to what you found with your own experimentation.
- c. Use the rule that you found in Part b to write and solve equations and inequalities matching these questions about the pumpkin shot.
 - i. When was the pumpkin 60 feet above the ground?
 - ii. For which time(s) was the pumpkin at least 60 feet above the ground?
- d. Use the rule you found in Part b to answer the following questions.
 - i. What is your best estimate for the maximum height of the pumpkin?
 - ii. How do you know if you have a good estimate? When does the pumpkin reach that height?

Summarize the Mathematics

In this investigation, you used several strategies to find rules for quadratic functions that relate the position of flying objects to time in flight. You used those function rules and resulting tables and graphs to answer questions about the problem situations.

- a** How can the height from which an object is dropped or launched be seen:
 - i. In a table of *(time, height)* values?
 - ii. On a graph of height over time?
 - iii. In a rule of the form $h = h_0 + v_0t - 16t^2$ giving height as a function of time?
- b** How could you determine the initial upward velocity of a flying object from a rule in the form $h = h_0 + v_0t - 16t^2$ giving height as a function of time?
- c** What strategies can you use to answer questions about the height of a flying object over time?

Be prepared to share your ideas and strategies with others in your class.

✓ Check Your Understanding



In Game 3 of the 1970 NBA championship series, the L.A. Lakers were down by two points with three seconds left in the game. The ball was inbounded to Jerry West, whose image is silhouetted in today's NBA logo. He launched and made a miraculous shot from beyond midcourt, a distance of 60 feet, to send the game into overtime (there was no 3-point line at that time).

Through careful analysis of the game tape, one could determine the height at which Jerry West released the ball, as well as the amount of time that elapsed between the time the ball left his hands and the time the ball reached the basket.

This information could then be used to write a rule for the ball's height h in feet as a function of time in flight t in seconds.

- a.** Suppose the basketball left West's hands at a point 8 feet above the ground. What does that information tell about the rule giving h as a function of t ?
- b.** Suppose also that the basketball reached the basket (at a height of 10 feet) 2.5 seconds after it left West's hands. Use this information to determine the initial upward velocity of the basketball.
- c.** Write a rule giving h as a function of t .

- d. Use the function you developed in Part c to write and solve equations and inequalities to answer these questions about the basketball shot.
- At what other time(s) was the ball at the height of the rim (10 feet)?
 - For how long was the ball higher than 30 feet above the floor?
 - If the ball had missed the rim and backboard, when would it have hit the floor?
- e. What was the maximum height of the shot, and when did the ball reach that point?

Investigation 2 Golden Gate Quadratics

The quadratic functions that describe the rise and fall of flying pumpkins are examples of a larger family of relationships described by rules in the general form $y = ax^2 + bx + c$. The particular numerical values of the coefficients a and b and the constant c depend on problem conditions. As you work through this investigation, look for answers to these questions:

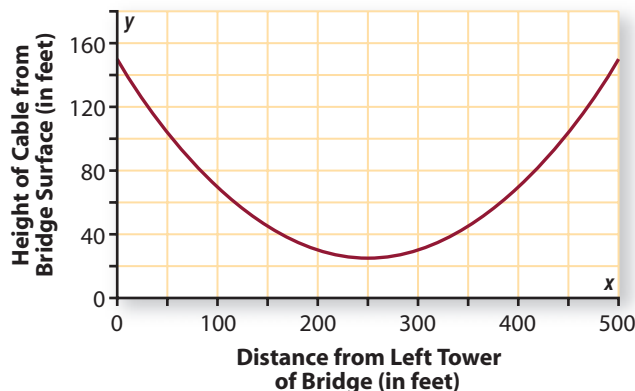
How can tables, graphs, and rules for quadratic functions be used to answer questions about the situations they represent?

What patterns of change appear in tables and graphs of quadratic functions?

Suspension Bridges Some of the longest bridges in the world are suspended from cables that hang in parabolic arcs between towers. One of the most famous suspension bridges is the Golden Gate Bridge in San Francisco, California.



If you think of one bridge tower as the y -axis of a coordinate system and the bridge surface as the x -axis, the shape of the main suspension cable is like the graph of a quadratic function. For example, if the function defining the curve of one suspension cable is $y = 0.002x^2 - x + 150$, where x and y are measured in feet, the graph will look like that on the top of the next page.



- 1 Use the function $y = 0.002x^2 - x + 150$ to answer the following questions.
- What is the approximate height (from the bridge surface) of the towers from which the cable is suspended?
 - What is the shortest distance from the cable to the bridge surface, and where does it occur?
 - For what interval(s) is the suspension cable at least 75 feet above the bridge surface?
 - Recall that the height function for dropping pumpkins was $h = 100 - 16t^2$ and for a basketball long shot was $h = 8 + 40.8t - 16t^2$.
 - How is the graph of the suspension cable function similar to and different from the graphs of these two functions?
 - How is the rule of the suspension cable function similar to and how is it different from the rules of these two functions?



Fundraising In 1996, the first Tibetan Freedom Concert, regarded by many as the single greatest cultural event in modern rock history, took place in Golden Gate Park in San Francisco. This was the first in a series of benefit concerts organized by the Milarepa Fund to raise awareness about nonviolence and the Tibetan struggle for freedom, as well as to encourage youth activism.

The primary goal for the Tibetan Freedom Concerts was to raise awareness, not money. However, careful planning was needed to ensure that the event would reach a large audience and that it would not *lose* money. The profit from

any event will be the difference between income and operating expenses.

- 2 As organizers planned for the event, they had many variables to consider.
- What factors will affect the number of tickets sold for the event?
 - What kinds of expenses will reduce profit from tickets sales, and how will those expenses depend on the number of people who buy tickets and attend?

3 Suppose that a group of students decided to organize a local concert to raise awareness and funds for the Tibetan struggle, and that planning for the concert led to this information:

- The relationship between number of tickets sold s and ticket price x in dollars can be approximated by the linear function $s = 4,000 - 250x$.
 - Expenses for promoting and operating the concert will include \$1,000 for advertising, \$3,000 for pavilion rental, \$1,500 for security, and \$2,000 for catering and event T-shirts for volunteer staff and band members.
- a. Find a function that can be used to predict income I for any ticket price x .
 - b. Find a function that can be used to predict profit P for any ticket price x .
 - c. How do predicted income and profit change as the concert organizers consider ticket prices ranging from \$1 to \$20? How are those patterns of change shown in graphs of the income and profit functions?
 - d. What ticket price(s) seem likely to give maximum income and maximum profit for the concert? What are those maximum income and profit values? How many tickets will be sold at the price(s) that maximize income and profit?
 - e. If event planners are more interested in attracting a large audience without *losing* money on the event than in maximizing profit, what range of ticket prices should they consider? Explain your reasoning.

4 The **break-even point** is the ticket price for which the event's income will equal expenses. Another way to think of the break-even point is the ticket price when profit is \$0.

- a. Write and solve an equation that can be used to find the break-even ticket price for this particular planned concert.
- b. Write and solve an inequality that can be used to find ticket prices for which the planned concert will make a positive profit.
- c. Write and solve an inequality that can be used to find ticket prices for which the planned concert will lose money.

5 What similarities and differences do you see in tables, graphs, and rules of the functions relating number of tickets sold, income, and profit to the proposed ticket price?

Summarize the Mathematics

In this investigation, you used several strategies to find and use rules for quadratic functions in different problem situations.

- a** How are patterns in the tables of values, graphs, and rules of the quadratic function examples in Investigations 1 and 2 similar to each other, and how do they differ from each other?
- b** How are tables, graphs, and rules of quadratic functions similar to and different from those of other types of functions you have worked with in earlier studies?
- c** What strategies can you use to:
 - i. Produce functions that model problem conditions?
 - ii. Solve quadratic equations or inequalities?
 - iii. Find maximum or minimum values of quadratic functions?

Be prepared to share your ideas and strategies with others in your class.

Check Your Understanding

The physical forces that determine the shape of a suspension bridge and business factors that determine the graph of profit prospects for a concert apply to other situations as well. For example, the parabolic reflectors that are used to send and receive microwaves and sounds have shapes determined by quadratic functions.

Suppose that the profile of one such parabolic dish is given by the graph of $y = 0.05x^2 - 1.2x$, where dish width x and depth y are in feet.



- a.** Sketch a graph of the function $y = 0.05x^2 - 1.2x$ for $0 \leq x \leq 25$. Then write calculations, equations, and inequalities that would provide answers for parts i–iv. Use algebraic, numeric, or graphic reasoning strategies to find the answers, and label (with coordinates) the points on the graph corresponding to your answers.
 - i.** If the edge of the dish is represented by the points where $y = 0$, how wide is the dish?
 - ii.** What is the depth of the dish at points 6 feet in from the edge?
 - iii.** How far in from the edge will the depth of the dish be 2 feet?
 - iv.** How far in from the edge will the depth of the dish be at least 3 feet?
- b.** What is the maximum depth of the dish and at what distance from the edge will that occur? Label the point (with coordinates) on your graph of $y = 0.05x^2 - 1.2x$.

Investigation 3

Patterns in Tables, Graphs, and Rules

In your work on the problems of Investigations 1 and 2, you examined a variety of rules, tables, and graphs for quadratic functions. For example,

A pumpkin's height above ground (in feet) is given by $h = 100 - 16t^2$.

A suspension cable's height above a bridge surface (in feet) is given by $y = 0.002x^2 - x + 150$.

Income for a concert (in dollars) is given by $I = x(4,000 - 250x)$.

Profit for a concert (in dollars) is given by $P = x(4,000 - 250x) - 7,500$.

It turns out that the expressions used in all of these functions are equivalent to expressions in the general form $ax^2 + bx + c$. To solve problems involving quadratic functions, it helps to know how patterns in the expressions $ax^2 + bx + c$ are related to patterns in tables and graphs of the related quadratic functions $y = ax^2 + bx + c$. As you work on the following problems, look for answers to this question:

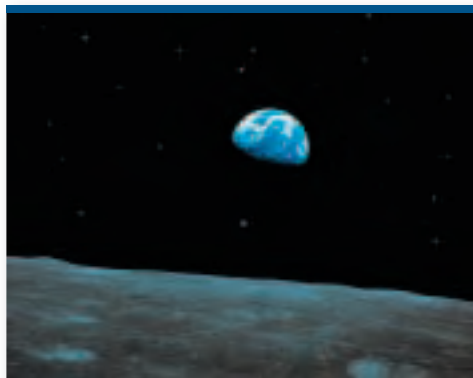
How are the values of a , b , and c related to patterns in the graphs and tables of values for quadratic functions $y = ax^2 + bx + c$?

To answer a question like this, it helps to use your calculator or CAS software to produce tables and graphs of many examples in which the coefficients a and b and the constant c are varied systematically. The problems of this investigation suggest ways you could do such explorations and some questions that might help in summarizing patterns you notice. Make informal notes of what you observe in the experiments and then share your ideas with your teacher and other students to formulate some general conclusions.

The Basic Quadratic Function When distance traveled by a falling pumpkin is measured in feet, the rule giving distance as a function of time is $d = 16t^2$. When such gravitational effects are studied on the Moon, the rule becomes $d = 2.6t^2$. When distance is measured in meters, the rule is $d = 4.9t^2$ on the Earth and $d = 0.8t^2$ on the Moon.

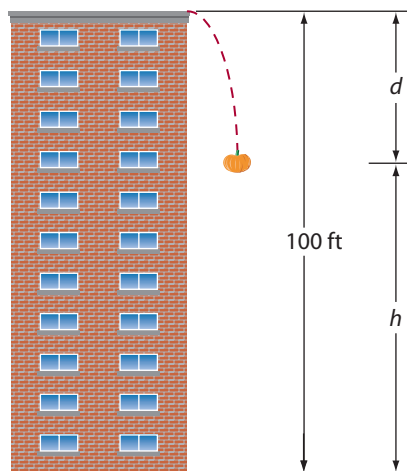
These are all examples of the simplest quadratic functions—those defined by rules in the form $y = ax^2$.

How can you predict the shape and location of graphs of quadratic functions with rules in the form $y = ax^2$?



- 1 Study the tables and graphs produced by such functions for several *positive* values of a . For example, you might start by comparing tables and graphs of $y = x^2$, $y = 2x^2$, and $y = 0.5x^2$ for $-10 \leq x \leq 10$.
 - a. What do all the graphs have in common? How about all the tables?
 - b. How is the pattern in a table or graph of $y = ax^2$ related to the value of the coefficient a when $a > 0$?
- 2 Next study the tables and graphs produced by such functions for several *negative* values of a . For example, you might start by comparing tables and graphs of $y = -x^2$, $y = -2x^2$, and $y = -0.5x^2$ for $-10 \leq x \leq 10$.
 - a. What do all these tables have in common? How about all the graphs?
 - b. How is the pattern in a table or graph of $y = ax^2$ related to the value of the coefficient a , when $a < 0$?
- 3 Now think about *why* the patterns in tables and graphs of functions $y = ax^2$ occur and *why* the coefficient a is helpful in predicting behavior of any particular quadratic in this form.
 - a. Consider first the functions $y = ax^2$ when $a > 0$.
 - i. Why are the values of y always greater than or equal to zero?
 - ii. Why are the graphs always symmetric curves with a minimum point $(0, 0)$?
 - b. Consider next the functions $y = ax^2$ when $a < 0$.
 - i. Why are the values of y always less than or equal to zero?
 - ii. Why are the graphs always symmetric curves with a maximum point $(0, 0)$?

Adding a Constant When you designed a quadratic function to model position of a pumpkin at various times after it is dropped from a 100-foot tall building, the rules that made sense were $y = 100 - 16t^2$ or $y = -16t^2 + 100$.



If the building from which pumpkins are to be dropped were taller or shorter, the rules might be $y = -16t^2 + 150$ or $y = -16t^2 + 60$.

These are all examples of another family of quadratic functions—those defined by rules in the form $y = ax^2 + c$.

How can you predict the shape and location of graphs of quadratic functions with rules in the form $y = ax^2 + c$?

- 4** Study tables and graphs produced by such functions for several combinations of positive and negative values of a and c . You might start by comparing these sets of functions:

Set 1	Set 2	Set 3
$y = x^2$	$y = -x^2$	$y = 2x^2$
$y = x^2 + 3$	$y = -x^2 + 5$	$y = 2x^2 + 1$
$y = x^2 - 4$	$y = -x^2 - 1$	$y = 2x^2 - 3$

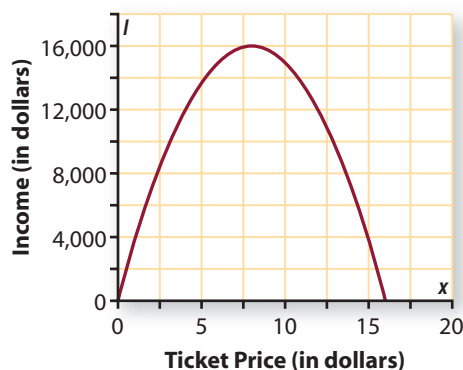
- How is the graph of $y = ax^2 + c$ related to the graph of $y = ax^2$?
- How is the relationship between $y = ax^2 + c$ and $y = ax^2$ shown in tables of (x, y) values for the functions?
- What are the values of $y = ax^2 + c$ and $y = ax^2$ when $x = 0$? How do these results help to explain the patterns relating the types of quadratics that you described in Parts a and b?

Factored and Expanded Forms When you studied problems about income from an amusement park bungee jump and promotion of a concert, you looked at functions relating income to ticket price. The resulting income rules had similar forms:

Bungee Jump: $I = p(50 - p)$



Concert Promotion: $I = x(4,000 - 250x)$



Just as you did with linear expressions in Unit 3, you can apply properties of numbers and operations to rewrite these rules in equivalent *expanded* form.

- 5 To rewrite the rule $I = p(50 - p)$, a student at Sauk Prairie High School reasoned like this:

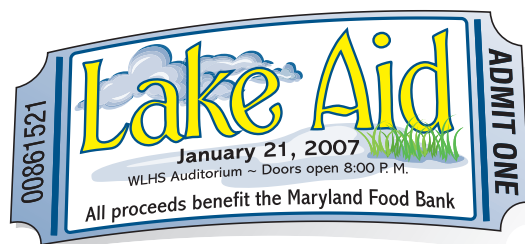
Applying the distributive property, $p(50 - p) = 50p - p^2$.

Rearranging terms, $50p - p^2 = -p^2 + 50p$.

So $I = -p^2 + 50p$, showing that income is a quadratic function of ticket price.

- Use similar ideas to rewrite $x(4,000 - 250x)$ in an equivalent expanded form.
- Study graphs of the two income functions: $I = p(50 - p)$ and $I = x(4,000 - 250x)$. In each case, find coordinates of:
 - the y-intercept,
 - the x-intercepts, and
 - the maximum point.
- How could you find these special points in Part b by analyzing the symbolic function rules in factored and/or expanded forms?
- The Sauk Prairie student made the following observations. How do you think the student arrived at those ideas? Do you agree with them? If not, explain why not.
 - It is easiest to find the y-intercept from the expanded form $-p^2 + 50p$.
 - It is easiest to find x-intercepts of the income function graph from the factored form $p(50 - p)$.
 - It is easiest to find the maximum point on the income graph from the x-intercepts.

- 6 The planning committee for Lake Aid, an annual benefit talent show at Wilde Lake High School, surveyed students to see how much they would be willing to pay for tickets. Suppose the committee developed the function $I = -75p^2 + 950p$ to estimate income I in dollars for various ticket prices p in dollars. Use the patterns you observed in Problem 5 to help answer the following questions.



- Write the function for income using an equivalent factored form of the expression given. What information is shown well in the factored form that is not shown as well in the expanded form?
- For what ticket prices does the committee expect an income of \$0?
- What ticket price will generate the greatest income? How much income is expected at that ticket price?
- Use your answers to Parts b and c to sketch a graph of $I = -75p^2 + 950p$.

Adding a Linear Term The income functions you studied in Problems 5 and 6 are examples of another family of quadratics, those with rules in the form $y = ax^2 + bx$.

How can you predict the shape and location of graphs of quadratic functions with rules in the form $y = ax^2 + bx$?

- 7** Study tables and graphs produced by such functions for several combinations of positive and negative values of a and b . You might start by comparing graphs of the following sets of functions:

Set 1	Set 2	Set 3
$y = x^2$	$y = -x^2$	$y = 2x^2$
$y = x^2 + 4x$	$y = -x^2 + 5x$	$y = 2x^2 + 6x$
$y = x^2 - 4x$	$y = -x^2 - 5x$	$y = 2x^2 - 6x$

Look at graphs of the functions given above to see if you can find patterns that relate the values of a and b in the rules $y = ax^2 + bx$ to location of the features below. It might help to think about the functions using the equivalent factored form, $x(ax + b)$.

- y -intercepts
- x -intercepts
- maximum or minimum point

Putting Things Together The graphs of all quadratic functions are curves called *parabolas*. In work on Problems 1–7, you have learned how to predict the patterns in graphs for three special types of quadratic functions: $y = ax^2$, $y = ax^2 + c$, and $y = ax^2 + bx$. You can use what you know about these quadratic functions to reason about graphs produced by functions when the coefficients a and b and the constant c are not zero.

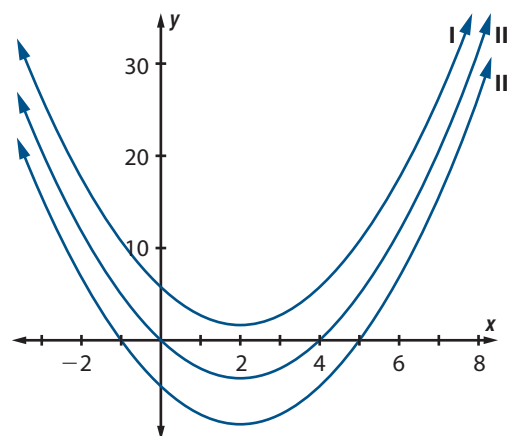
- 8** Explore the following examples and look for explanations of the patterns observed.

- a.** The diagram at the right gives graphs for three of the four quadratic functions below.

$$\begin{aligned} y &= x^2 - 4x \\ y &= x^2 - 4x + 6 \\ y &= -x^2 - 4x \\ y &= x^2 - 4x - 5 \end{aligned}$$

Without using graphing technology:

- Determine the function with the graph that is missing on the diagram.
- Match the remaining functions to their graphs, and be prepared to explain your reasoning.



- b. Without using graphing technology, sketch the pattern of graphs you would expect for the next set of quadratic functions. Explain your reasoning in making the sketch. Then check your ideas with the help of technology.

$$y = x^2 + 4x$$

$$y = x^2 + 4x - 6$$

$$y = x^2 + 4x + 5$$

- c. How would a sketch showing graphs of the following functions be similar to and different from those in Parts a and b? Explain your reasoning. Then check your ideas with the help of technology.

$$y = -x^2 + 4x$$

$$y = -x^2 + 4x - 6$$

$$y = -x^2 + 4x + 5$$

- d. How can properties of the special quadratic functions $y = ax^2$, $y = ax^2 + c$, and $y = ax^2 + bx$ help in reasoning about shape and location of graphs for functions in the form $y = ax^2 + bx + c$?

Summarize the Mathematics

In this investigation, you discovered some facts about the ways that patterns in tables and graphs of quadratic functions $y = ax^2 + bx + c$ ($a \neq 0$) are determined by the values of a , b , and c .

- a What does the sign of a tell about the patterns of change and graphs of quadratic functions given by rules in the form $y = ax^2$? What does the absolute value of a tell you?
- b How are the patterns of change and graphs of quadratic functions given by rules like $y = ax^2 + c$ related to those of the basic quadratic function $y = ax^2$? What does the value of c tell about the graph?
- c How are the graphs of functions defined by rules like $y = ax^2 + bx$ ($b \neq 0$) different from those of functions with rules like $y = ax^2$? What does the value of b tell about the graph?
- d How can you use what you know about quadratic functions with rules $y = ax^2$, $y = ax^2 + c$, and $y = ax^2 + bx$ to predict the shape and location of graphs for quadratic functions with rules $y = ax^2 + bx + c$ in which none of a , b , or c is 0?

Be prepared to share your ideas with the class.

Check Your Understanding

Use what you know about the relationship between rules and graphs for quadratic functions to match the functions with their graphs. Graphs were all produced with the same windows.

Rule I $y = x^2 + 2$

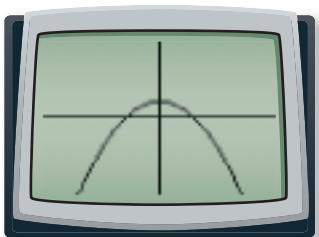
Rule II $y = x^2 - 5x + 2$

Rule III $y = -x^2 + 2$

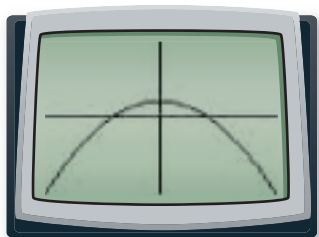
Rule IV $y = -0.5x^2 + 2$

Rule V $y = x^2 + 5x + 2$

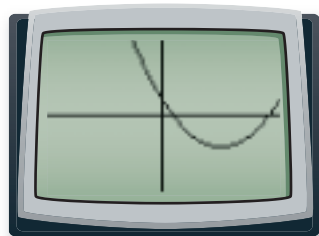
A



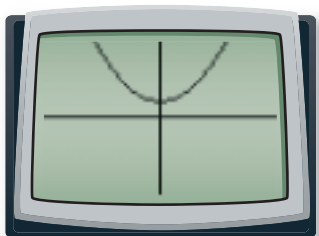
B



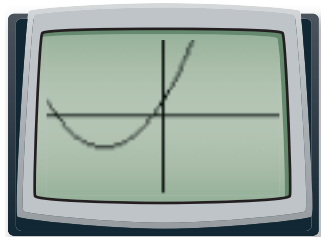
C



D



E



On Your Own

Applications

- 1** A first-time diver was a bit nervous about his first dive at a swimming pool. To ease his worries about hitting the water after a fall of 15 feet, he decided to push a tennis ball off the edge of the platform to see the effect of landing in the water.

 - a.** What rule shows how the ball's height above the water h is related to elapsed time in the dive t ?
 - b.** Estimate the time it will take the ball to hit the water.
- 2** Katie, a goalie for Riverside High School's soccer team, needs to get the ball downfield to her teammates on the offensive end of the field. She punts the ball from a point 2 feet above the ground with an initial upward velocity of 40 feet per second.

 - a.** Write a function rule that relates the ball's height above the field h to its time in the air t .
 - b.** Use this function rule to estimate the time when the ball will hit the ground.
 - c.** Suppose Katie were to kick the ball right off the ground with the same initial upward velocity. Do you think the ball would be in the air the same amount of time, for more time, or for less time? Check your thinking.
- 3** The opening of the cannon pictured at the left is 16 feet above the ground. The daredevil, who is shot out of the cannon, reaches a maximum height of 55 feet after about 1.56 seconds and hits a net that is 9.5 feet off the ground after 3.25 seconds. Use this information to answer the following questions.

 - a.** Write a rule that relates the daredevil's height above the ground h at a time t seconds after the cannon is fired.
 - b.** At what upward velocity is the daredevil shot from the cannon?
 - c.** If, for some unfortunate reason, the net slipped to the ground at the firing of the cannon, when would the daredevil hit the ground?
- 4** When a punkin' chunker launches a pumpkin, the goal is long distance, not height. Suppose the relationship between horizontal distance d (in feet) and time t (in seconds) is given by the function rule $d = 70t$, when the height is given by $h = 20 + 50t - 16t^2$.

 - a.** How long will the pumpkin be in the air?
 - b.** How far will the pumpkin travel from the chunker by the time it hits the ground?



- c. When will the pumpkin reach its maximum height, and what will that height be?
- d. How far from the chunker will the pumpkin be (horizontally) when it reaches its maximum height?

- 5 Imagine you are in charge of constructing a two-tower suspension bridge over the Potlatch River. You have planned that the curve of the main suspension cables can be modeled by the function $y = 0.004x^2 - x + 80$, where y represents height of the cable above the bridge surface and x represents distance along the bridge surface from one tower toward the other. The values of x and y are measured in feet.



- a. What is the approximate height (from the bridge surface) of each tower from which the cable is suspended?
- b. What is the shortest distance from the cable to the bridge surface and where does it occur?
- c. At which points is the suspension cable at least 50 feet above the bridge surface? Write an inequality that represents this question and express the solution as an inequality.

- 6 One formula used by highway safety engineers relates minimum stopping distance d in feet to vehicle speed s in miles per hour with the rule $d = 0.05s^2 + 1.1s$.
- a. Create a table of sample (*speed, stopping distance*) values for a reasonable range of speeds. Plot the sample (*speed, stopping distance*) values on a coordinate graph. Then describe how stopping distance changes as speed increases.
 - b. Use the stopping distance function to answer the following questions.
 - i. What is the approximate stopping distance for a car traveling 60 miles per hour?
 - ii. If a car stopped in 120 feet, what is the fastest it could have been traveling when the driver first noticed the need to stop?
 - c. Estimate solutions for the following quadratic equations and explain what each solution tells about stopping distance and speed.
 - i. $180 = 0.05s^2 + 1.1s$
 - ii. $95 = 0.05s^2 + 1.1s$

- 7 Use what you know about the connection between rules and graphs for quadratic functions to match the given functions with their graphs that appear below. Each graph is shown in the standard viewing window ($-10 \leq x \leq 10$ and $-10 \leq y \leq 10$).

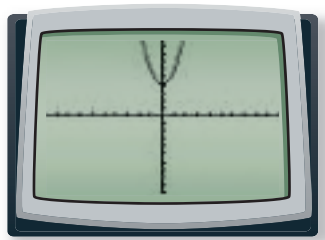
Rule I $y = x^2 - 4$

Rule II $y = 2x^2 + 4$

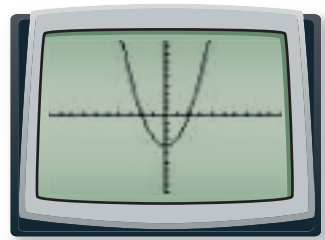
Rule III $y = -x^2 + 2x + 4$

Rule IV $y = -0.5x^2 + 4$

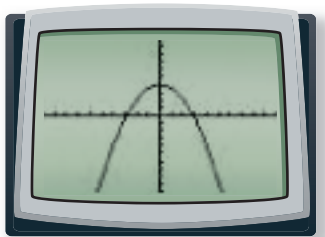
A



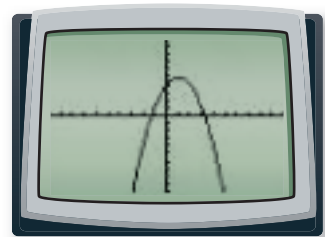
B



C



D



- 8 In Applications Tasks 1, 2, 5, and 6, you worked with several different quadratic functions. The function rules are restated in Parts a–d below. For each function, explain what you can learn about the shape and location of its graph by looking at the coefficients and constant term in the rule.

a. $h = 15 - 16t^2$

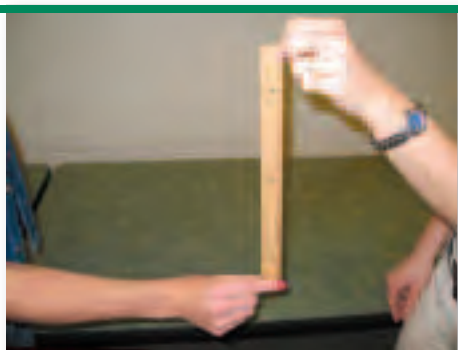
b. $h = 2 + 40t - 16t^2$

c. $y = 0.004x^2 - x + 80$

d. $d = 0.05s^2 + 1.1s$

Connections

- 9 The following experiment can be used to measure a person's *reaction time*, the amount of time it takes a person to react to something he or she sees.



Hold a ruler at the end that reads 12 inches and let it hang down. Have the subject hold his or her thumb and forefinger opposite the 0-inch mark without touching the ruler. Tell your subject that you will drop the ruler within the next 10 seconds and that he or she is supposed to grasp the ruler as quickly as possible after it is dropped.

The spot on the ruler where it is caught indicates the distance that the ruler dropped.

- a. What function describes the distance d in feet that the ruler has fallen after t seconds?
- b. Use what you know about the relationship between feet and inches and your function from Part a to estimate the reaction time of a person who grasps the ruler at the 4-inch mark.
- c. Conduct this experiment several times and estimate the reaction times of your subjects.

10 Consider some other familiar measurement formulas.

- a. Match the formulas A–D to the measurement calculations they express:

I volume of a cube	A $y = s^2$
II surface area of a cube	B $y = s^3$
III area of a square	C $y = 4s$
IV perimeter of a square	D $y = 6s^2$

- b. Which of the formulas from Part a are those of quadratic functions?

11 The formula $A = \pi r^2$ shows how to calculate the area of a circle from its radius. You can also think about this formula as a quadratic function.

- a. With respect to the general form of a quadratic function $y = ax^2 + bx + c$, what are the a , b , and c values for the area-of-a-circle function?
- b. For $r > 0$, how does the shape of the graph $A = \pi r^2$ compare to that of $A = r^2$?
- c. If the radius of a circle is 6 cm, what is its area?
- d. If the area of a circle is about 154 cm^2 , what is the approximate radius of the circle?

12 Sketch graphs of the functions $y = 2x$, $y = x^2$, and $y = 2^x$ for $0 \leq x \leq 5$.

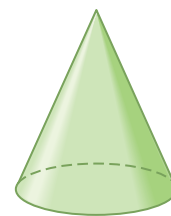
- a. In what ways are the graphs similar to each other?
- b. In what ways do the graphs differ from each other?
- c. The values of y for the three functions when x is between 0 and 1 show that $x^2 < 2x < 2^x$. Compare the values of y for the three graphs for the intervals for x below.
 - i. between 1 and 2
 - ii. between 2 and 3
 - iii. between 3 and 4
 - iv. greater than 4

13 Suppose that a pumpkin is dropped from an airplane flying about 5,280 feet above the ground (one mile up in the air). The function $h = 5,280 - 16t^2$ can be used to predict the height of that pumpkin at a point t seconds after it is dropped. But this mathematical model ignores the effects of air resistance.

- How would you expect a height function that does account for air resistance to be different from the function $h = 5,280 - 16t^2$ that ignores those effects?
- The speed of the falling pumpkin at a time t seconds after it is dropped can be predicted by the function $s_1 = 32t$, if you ignore air resistance. If air resistance is considered, the function $s_2 = 120(1 - 0.74^t)$ will better represent the relationship between speed and time.
 - Make tables of *(time, speed)* values for each function s_1 and s_2 with values of t from 0 to 10 seconds.
 - Sketch graphs showing the patterns of change in speed implied by the two functions.
 - Describe similarities and differences in patterns of change predicted by the two *(time, speed)* functions.
- Air resistance on the falling pumpkin causes the speed of descent to approach a limit called *terminal velocity*. Explore the pattern of *(time, speed)* values for the function $s_2 = 120(1 - 0.74^t)$ for larger and larger values of t to see if you can discover the terminal velocity implied by that speed function.

14 Graphs of quadratic functions are curves called parabolas. Parabolas and other curves can also be viewed as cross sections of a cone—called *conic sections*.

- Describe how you could position a plane intersecting a cone so that the cross section is a parabola.
- How could you position a plane intersecting a cone so that the cross section is a circle?
- What other curve(s) are formed by a plane intersecting a cone? Illustrate your answer.

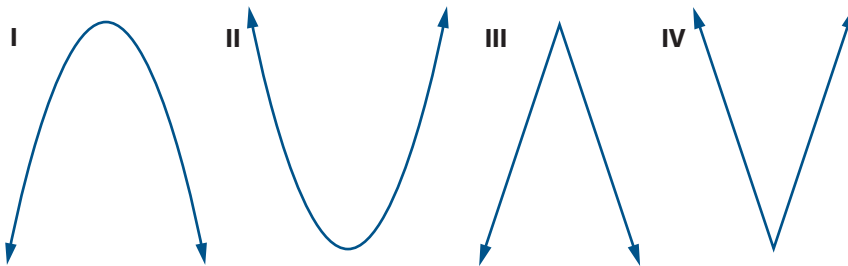


Reflections

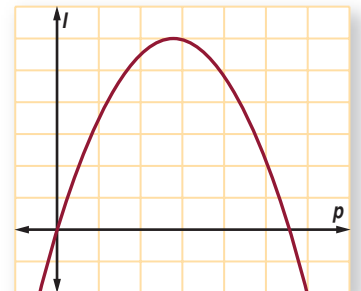
- 15** Suppose that a skateboard rider travels from the top of one side to the top of the other side on a half-pipe ramp.



Which of the following graphs is the best model for the relationship between the rider's speed and distance traveled? Explain your choice.



- 16** In several problems about the relation between income and price for a business venture, you worked with quadratic functions that have graphs like the one shown to the right.
- How would you describe the pattern of change in predicted income as ticket price increases?
 - Why is that general pattern reasonable in a wide variety of business situations?
- 17** A student first studying quadratic functions had the idea that in the rule $y = ax^2 + bx + c$, the value of b should tell the slope of the graph and c should tell the y -intercept. Do you agree? How could you use a graph of the function $y = x^2 + 2x + 3$ and other reasoning to support or dispute the student's idea?
- 18** All linear functions can be described by rules in the form $y = a + bx$. All exponential functions can be described by rules in the form $y = a(b^x)$. All quadratic functions can be described by rules in the form $y = ax^2 + bx + c$. The letters a , b , and c take on specific values in each of the three function forms. What information about each type of function can be learned from the values taken on by the letters a , b , and c ?



Extensions

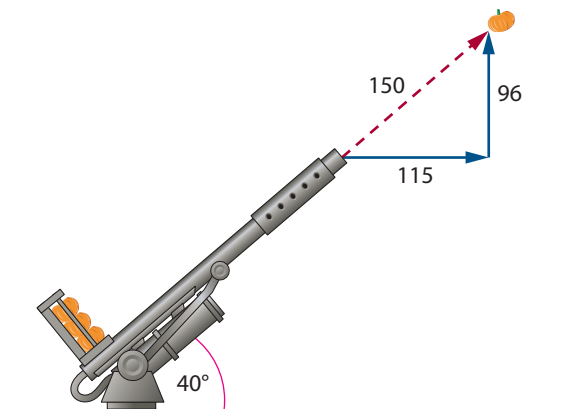
- 19** For anything that moves, *average speed* can be calculated by dividing the total distance traveled by the total time taken to travel that distance.

For example, a diver who falls from a 35-foot platform in about 1.5 seconds has an average speed of $\frac{35}{1.5}$, or about 23.3 feet per second. That diver will not be falling at that average speed throughout the dive.

- If a diver falls from 35 feet to approximately 31 feet in the first 0.5 seconds of a dive, what estimate of speed would seem reasonable for the diver midway through that time interval—that is, how fast might the diver be moving at 0.25 seconds?
- The relation between height above the water and the diver's time in flight can be described by the function $h = 35 - 16t^2$, if time is measured in seconds and distance in feet. Use that function rule to make a table of *(time, height)* data and then estimate the diver's speed at 6 points using your data. Make a table and a graph of the *(time, speed)* estimates.
- What do the patterns in *(time, speed)* data and the graph tell you about the diver's speed on the way to the water?
- About how fast is the diver traveling when he hits the water?
- Write a rule for speed s as a function of time t that seems to fit the data in your table and graph. Use your calculator or computer software to check the function against the data in Part b.



- 20** When a pumpkin is shot from an air cannon chunker, its motion has two components—vertical and horizontal. Suppose that a pumpkin is shot at an angle of 40° with initial velocity of 150 feet per second and initial height 30 feet. The vertical component of its velocity will be about 96 feet per second; the horizontal component of its velocity will be about 115 feet per second.



- a. What function gives the height h of the pumpkin shot at any time t seconds after it leaves the chunker?
- b. What function gives the horizontal distance d traveled by the pumpkin at any time t seconds after it leaves the chunker?
- c. Use the functions in Parts a and b to find the horizontal distance traveled by the pumpkin by the time it hits the ground.
- d. Rewrite the relation between time and distance in Part b to give time as a function of distance.
- e. Combine the rule giving time as a function of horizontal distance and the rule giving height as a function of time to write a function rule giving height as a function of horizontal distance. (*Hint*: Replace each occurrence of t by an equivalent expression involving d .)
- f. Use the function developed in Part e to estimate the distance traveled by the pumpkin when it hits the ground. Then compare the result obtained in this way to your answer to Part c.

21 You may have heard of *terminal velocity* in connection with skydiving. Scientific principles predict that a function like $h_2 = 5,680 - 120t - 400(0.74^t)$ will predict the height of a pumpkin (in feet) at any time t seconds after it is dropped from an airplane flying at an altitude of one mile. This function (in contrast to the more familiar $h_1 = 5,280 - 16t^2$) accounts for the slowing effect of air resistance on the falling pumpkin.

- a. Use your calculator to produce a table showing predictions for height of the pumpkin using the functions h_1 and h_2 for times from 0 to 30 seconds. Record the data for times $t = 0, 5, 10, 15, 20, 25$, and 30 seconds. Describe the patterns of change in height over time that are shown in the *(time, height)* values of the two functions.
- b. Extend your table of *(time, height)* values to a point that gives estimates of the time it takes for the pumpkin to hit the ground.
- c. Study the patterns of change in height for the last 10 seconds before the pumpkin hits the ground. Explain how the pattern of change in height for function h_2 illustrates the notion of terminal velocity that you explored in Connections Task 13 Part c.

22 Consider the two functions $y = 2^x$ and $y = x^2$.

- a. How are the graphs of these two functions alike and how are they different?
- b. How many solutions do you expect for the equation $2^x = x^2$? Explain your reasoning.
- c. Estimate the solution(s) of the equation $2^x = x^2$ as accurately as possible. Explain or show how you estimated the solution(s).

23 Compare the quadratic function $y = x^2$ and the *absolute value function*, $y = |x|$.

- a. Sketch graphs of these two functions and describe ways that they are similar and ways that they are different.

- b. Find solutions for $x^2 = |x|$ by reasoning with the symbols themselves and then label the graph points representing the solutions with their coordinates.
- c. Find the value(s) of x for which $x^2 > |x|$ and for which $x^2 < |x|$. Then indicate points on the graph representing the coordinates of those solutions.

24 Consider the quadratic functions defined by these rules:

Rule I $y = 3x^2 - 5x + 9$

Rule II $y = 1.5x^2 - 5x + 9$

Rule III $y = 3x^2 + 4x - 23$

Rule IV $y = x^2 - 5x - 23$

- a. Examine graphs and/or tables of the functions to determine which of the functions have values that are relatively close to each other for large values of x .
- b. Based on your findings in Part a, which coefficient (a or b) in the quadratic standard form determines **right end behavior**—the values of y for large positive values of x ?
- c. Can the same be said for **left end behavior** (which you may have guessed is for negative values of x with large absolute values)?

25 A computer spreadsheet can be a useful tool for exploring the effect of each coefficient and the constant term on the pattern of change of a quadratic function with rule $y = ax^2 + bx + c$. For example, the table below was produced with a spreadsheet to study $y = 2x^2 - 5x + 7$.

Quadratic Patterns.xls							
	A	B	C	D	E	F	G
1	x	ax^2	bx	c	$ax^2 + bx + c$	$a =$	2
2	-5	50	25	7	82	$b =$	-5
3	-4	32	20	7	59	$c =$	7
4	-3	18	15	7	40		
5	-2	8	10	7	25		
6	-1	2	5	7	14		
7	0	0	0	7	7		
8	1	2	-5	7	4		
9	2	8	-10	7	5		
10	3	18	-15	7	10		
11	4	32	-20	7	19		
12	5	50	-25	7	32		

- a. What numerical and formula entries and other spreadsheet techniques are needed to produce the x values in cells A2–A12?
- b. What spreadsheet formulas can be used to produce entries in cells B2, C2, D2, and E2?
- c. What formulas will appear in cells B3, C3, D3, and E3?
- d. How could the spreadsheet be modified to study the function $y = x^2 + 3x - 5$?

- 26 Important questions about quadratic functions sometimes require solving inequalities like $10 > x^2 + 2x - 5$ or $-2x^2 + 6x \geq -8$.
- What is the goal of the process in each case?
 - How can you use graphs to solve the inequalities?
 - How can you use tables of values to solve the inequalities?
 - How many solutions would you expect for a quadratic inequality?

Review

- 27 At many basketball games, there is a popular half-time contest to see if a fan can make a half-court shot. Some of these contests offer prizes of up to \$1,000,000! You may wonder how schools and other organizations could afford such payouts. In many cases, the organization offering the contest has purchased insurance to cover the costs, in the rare event that someone happens to make the shot. Imagine you've decided to start *Notgonnahappen Insurance* and your company will specialize in insuring \$1 million prizes.

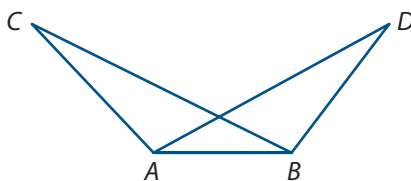
- If you charge organizations \$2,000 per contest for insurance, how many contests would you need to insure to cover the cost of a single event in which a contestant makes a million dollar half-court shot?
- Suppose that for the first \$1,000,000 you collect in insurance fees, there are no payouts. You decide to invest this money in a savings account for future contests.
 - If the account earns 4% interest compounded annually, how much would the account be worth in one year if no deposits or withdrawals were made?
 - Write two rules for calculating the account balance b at the end of t years—one using the *NOW-NEXT* approach and the other " $b = \dots$."
 - How much would the account be worth in 10 years if no deposits or withdrawals were made?



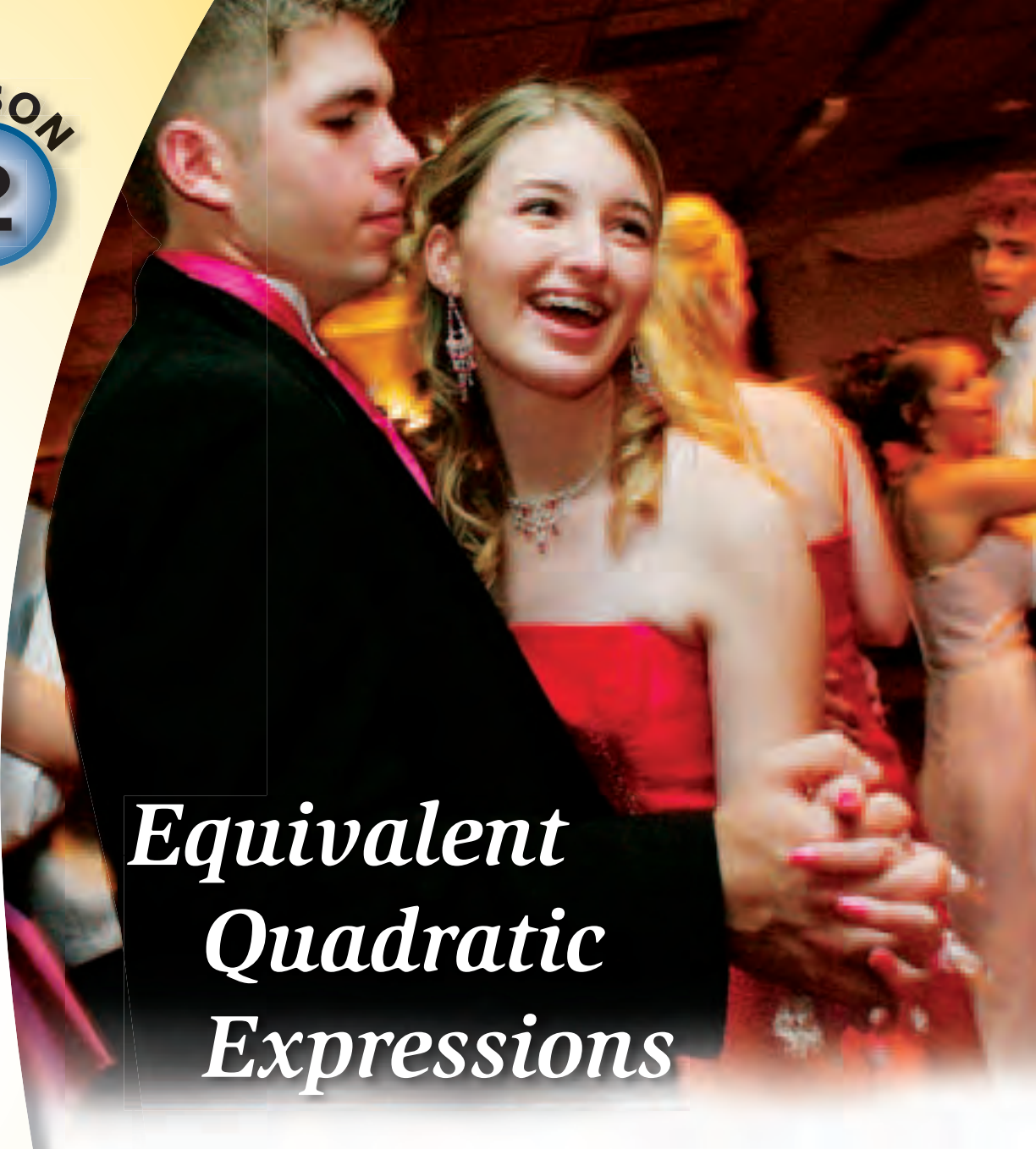
- 28 Write an equation for the line that:
- Contains the points $(0, 4)$ and $(5, -3)$.
 - Contains the points $(-2, 3)$ and $(-5, 6)$.
 - Contains the point $(7, 5)$ and has slope $\frac{2}{3}$.
 - Contains the point $(-2, 5)$ and is parallel to the line $y = 1.5x + 6$.

- 29 In the diagram at the right, $\overline{AC} \cong \overline{BD}$ and $\angle BAC \cong \angle ABD$. Using only this information:

- Explain why $\triangle ABC \cong \triangle BAD$.
- Explain why $\overline{AD} \cong \overline{BC}$.



- 30** Write each of the following exponential expressions in the form 5^x for some integer x .
- a. $(5^3)(5^4)$
- b. $5^7 \div 5^3$
- c. $(5^2)^3$
- d. $4(5^3) + 5^3$
- 31** Suppose that the 15 numbers below are a sample of fares (in dollars) collected by drivers for *Fast Eddy's Taxi* company from trips on one typical day.
- 13, 23, 20, 22, 27, 21, 29, 31, 12, 10, 11, 21, 5, 19, 36
- a. What are the mean and median of this sample of fares collected?
- b. What are the range and the standard deviation of the sample of fares?
- c. If there is a \$2 local government tax on each taxi fare, what are the mean, median, range, and standard deviation of the sample of fares after taxes have been deducted?
- Fast Eddy's Taxi* allows each driver to keep 70% of each after-tax fare as her or his pay.
- d. What are the mean and median driver earnings from the sample of trips?
- e. What are the range and standard deviation of driver earnings from the sample of trips?
- 32** In the *Patterns in Shape* unit, you revisited how to calculate areas and volumes of shapes.
- a. What is the area of an equilateral triangle with side lengths of 10 units?
- b. What is the area of a regular hexagon with side lengths of 10 units?
- c. What additional information do you need to find the volume of a prism with a regular hexagonal base that has side lengths of 10 units?
- d. How would the volumes of a prism and a pyramid compare if they had the same hexagonal base and same height?
- 33** Find the prime factorization of each number.
- a. 28
- b. 105
- c. 72
- d. 297
- 34** Rewrite each expression in a simpler equivalent form by first using the distributive property and then combining like terms.
- a. $6x(3x - 5) + 12x$
- b. $22 - 2(15 - 4x)$
- c. $\frac{1}{2}(12x + 7) + \frac{2}{3}(9 - 15x)$
- d. $15 - (3x - 8) + 5x(6 + 3x)$
- 35** Sketch a cylinder with radius r and height h . What will change the volume of the cylinder more, doubling the radius or doubling the height? Explain your reasoning.

A young man and woman are shown in formal attire, likely at a school party. The man is wearing a black tuxedo with a pink shirt, and the woman is wearing a red strapless dress. They are both smiling and looking towards the right. The background is slightly blurred, showing other people in formal wear.

LESSON 2

Equivalent Quadratic Expressions

When the freshman class officers at Sturgis High School were making plans for the annual end-of-year class party, they had a number of variables to consider:

- the number of students purchasing tickets and attending the party
- the price charged for tickets
- expenses, including food, a DJ, security, and clean-up

A survey of class members showed that the number of students attending the party would depend on the price charged for tickets. Survey data suggested a linear model relating price x and number of tickets sold n : $n = 200 - 10x$.

The students responsible for pricing food estimated an expense of \$5 per student.

The students responsible for getting a DJ reported that the person they wanted would charge \$150 for the event.

The school principal said that costs of security and clean-up by school crews would add another \$100 to the cost of the party.

Think About This Situation

Think about how you could use the above information to set a price that would guarantee the freshman class would not lose money on the party.

- a** What party profit could be expected if ticket price is set at \$5 per person? What if the ticket price is set at \$10 per person?
- b** How could the given information be combined to figure out a price that would allow the class to break even or maybe even make some profit for the class treasury?

In this lesson, you will explore ways to develop expressions for functions that model quadratic patterns of change and to write and reason with equivalent forms of those expressions.

Investigation 1 Finding Expressions for Quadratic Patterns

When the freshman class officers at Sturgis High School listed all the variables to be considered in planning their party, they disagreed about how to set a price that would guarantee a profitable operation. To help settle those arguments, they tried to find a single function showing how profit would depend on the ticket price. They came up with two different profit functions and wondered whether they were algebraically equivalent. As you work on the problems in this investigation, look for answers to these questions:

What strategies are useful in finding rules for quadratic functions?

In deciding whether two quadratic expressions are equivalent?

In deciding when one form of quadratic expression is more useful than another?

- 1 One way to discover functions that model problem conditions is to consider a variety of specific pairs of (x, y) values and look for a pattern relating those values.
 - a. Use the information on page 491 to find ticket sales, income, costs, and profits for a sample of possible ticket prices. Record results in a table like this:

	Ticket Price (in \$)				
	0	5	10	15	20
Number of Tickets Sold	200	150			
Income (in \$)	0	750			
Food Cost (in \$)	1,000				
DJ Cost (in \$)	150				
Security/Cleanup Cost (in \$)	100				
Profit (in \$)	-1,250				

- b. Plot the sample (*ticket price, profit*) values and describe the kind of function that you would expect to model the data pattern well.
- 2 Ms. Parkhurst, one of the Sturgis mathematics teachers, suggested another way to find a profit function that considers all factors. Check each step of her reasoning and explain why it is correct.
 - (1) Since the number of tickets sold n is related to the ticket price x by the linear function $n = 200 - 10x$, income from ticket sales will be related to ticket price by the function $I = x(200 - 10x)$.
 - (2) The cost c for food is related to the number of tickets sold n by the function $c = 5n$, so the cost for food will be related to the ticket price by the function $c = 5(200 - 10x)$.
 - (3) The costs for a DJ, security, and cleanup total \$250, regardless of the number of students who attend the party.
 - (4) So, the profit of the party can be predicted from the ticket price x using the function $P = x(200 - 10x) - 5(200 - 10x) - 250$.
- 3 Some students followed Ms. Parkhurst's reasoning up to the point where she said the profit function would be $P = x(200 - 10x) - 5(200 - 10x) - 250$. But they expected a quadratic function like $P = ax^2 + bx + c$.
 - a. Is the expression $x(200 - 10x) - 5(200 - 10x) - 250$ equivalent to an expression in the $ax^2 + bx + c$ form? If so, what is the expression in that form? If not, how do you know?
 - b. What are the advantages of expanded and simplified expressions in reasoning about the function relating party profit to ticket price?



- 4 Ms. Parkhurst's rule for the function relating party profit to ticket price involves products and sums of linear functions, but the result is a quadratic function. For each of the following pairs of linear functions:
- Graph the sum and describe the type of function that results from that operation. For example, for the functions in Part a, the sum is $y = (x + 2) + 0.5x$.
 - Graph the product and describe the type of function that results from that operation. For example, for the functions in Part a, the product is $y = (x + 2)(0.5x)$.
 - Write each sum and product in simpler equivalent form.

Be prepared to explain the reasoning you used to produce the equivalent function expressions.

- a. $y_1 = x + 2$ and $y_2 = 0.5x$ b. $y_1 = 2x - 3$ and $y_2 = -1.5x$
c. $y_1 = -3x$ and $y_2 = 5 - 0.5x$ d. $y_1 = x + 2$ and $y_2 = 2x + 1$

Summarize the Mathematics

In this investigation, you explored two ways to develop expressions for quadratic functions relating variables and ways to compare expressions to see if they are equivalent.

- What are the advantages of each strategy for developing rules for quadratic functions—looking for patterns in sample (x, y) data or using only reasoning about problem conditions?
- What does it mean to say that two algebraic expressions are equivalent?
- In what ways can you check to see whether two expressions are equivalent?
- Why might it be useful to write a quadratic expression in a different equivalent form?
- What graph and rule patterns would you expect from combining two linear functions by addition? By multiplication?

Be prepared to explain your ideas to the class.

✓ Check Your Understanding

Use your understanding of equivalent expressions to help complete the following tasks.

- Which of the following pairs of algebraic expressions are equivalent, and how do you know?
 - $x^2 + 5x$ and $x(x + 5)$
 - $m(100 - m) + 25$ and $-m^2 + 100m + 25$
 - $43 - 5(x - 10)$ and $33 - 5x$

- b. Which of these functions are linear, which are quadratic, which are neither, and how can you justify your conclusions algebraically?
- i. $J = (4p)(7p - 3)$
 - ii. $y = (3x + 2) - (2x - 4)$
 - iii. $d = (6t - 4) \div (2t)$

Investigation 2 Reasoning to Equivalent Expressions

In your work with linear functions, you learned that the form $y = mx + b$ was very useful for finding the slope and intercepts of graphs. However, you also found that linear functions sometimes arise in ways that make other equivalent expressions natural and informative. The work on party planning in Investigation 1 showed that the same thing can happen with quadratic relations.

It is relatively easy to do some informal checking to see if two given quadratic expressions might be equivalent—comparing graphs or entries in tables of values. But there are also some ways that properties of numbers and operations can be used to prove equivalence of quadratic expressions and to write any given expression in useful equivalent forms. As you complete the following problems, look for answers to this question:

What strategies can be used to transform quadratic expressions into useful equivalent forms?

One basic principle used again and again to produce equivalent expressions is the *Distributive Property of Multiplication over Addition (and Subtraction)*. It states that for any numbers a , b , and c , $a(b + c) = (ab) + (ac)$ and $a(b - c) = (ab) - (ac)$. For example, $5(x + 7) = 5x + 35$ and $5(x - 7) = 5x - 35$.

- 1 Use the distributive property to expand and combine like terms to write each of the following expressions in equivalent standard form $ax^2 + bx + c$. Be prepared to explain your reasoning in each case.
- a. $(3 + x)x$
 - b. $5x(4x - 11)$
 - c. $7x(11 - 4x)$
 - d. $7x(x + 2) - 19$
 - e. $-9(5 - 3x) + 7x(x + 4)$
 - f. $mx(x + n) + p$

- 2 Use the distributive property to write each of these quadratic expressions in equivalent form as a product of two linear factors. Be prepared to explain your reasoning in each case.

- a. $7x^2 - 11x$
- b. $12x + 4x^2$
- c. $-3x^2 - 9x$
- d. $ax^2 + bx$

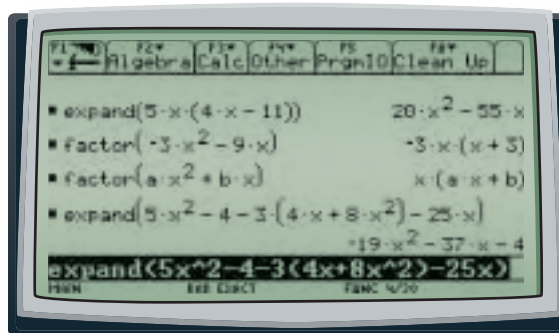
- 3 Sometimes you need to combine expanding, factoring, and rearrangement of terms in a quadratic expression in order to produce a simpler form that gives useful information. For example, the following work shows how to write a complex expression in simpler expanded and factored forms.

$$\begin{aligned} 5x(6x - 8) + 4x(2 - 3x) &= 30x^2 - 40x + 8x - 12x^2 \\ &= 18x^2 - 32x \\ &= 2x(9x - 16) \end{aligned}$$

Use what you know about ways of writing algebraic expressions in equivalent forms to produce simplest possible expanded and (where possible) factored forms of these expressions.

- a. $(14x^2 + 3x) - 7x(4 + x)$
- b. $-x + 4x(9 - 2x) + 3x^2$
- c. $5x(2x - 1) + 4x^2 - 2x$
- d. $(5x^2 - 4) - 3(4x + 8x^2) - 25x$

The distributive property is used many places in algebra to write expressions in equivalent forms. In fact, the operations of *expanding* and *factoring* expressions like those for quadratic functions are now built into computer algebra systems. The following screen shows several results that should agree with answers you got in Problems 1, 2, and 3.



You can use such a computer algebra tool to check your answers as you learn how expanding and factoring work and when you meet problems that require complicated symbol manipulation.

- 4** In some situations, a quadratic expression arises as the product of two linear expressions. In those cases, you can use the distributive property twice to expand the factored quadratic to standard form. Study the steps in these examples, and then apply similar reasoning to expand the expressions in Parts a–e.

Strategy 1

$$\begin{aligned}(x + 5)(x - 7) &= (x + 5)x - (x + 5)7 \\ &= x^2 + 5x - 7x - 35 \\ &= x^2 - 2x - 35\end{aligned}$$

Strategy 2

$$\begin{aligned}(x + 5)(x - 7) &= x(x - 7) + 5(x - 7) \\ &= x^2 - 7x + 5x - 35 \\ &= x^2 - 2x - 35\end{aligned}$$

- a. $(x + 5)(x + 6)$
- b. $(x - 3)(x + 9)$
- c. $(x + 10)(x - 10)$
- d. $(x - 5)(x + 1)$
- e. $(x + a)(x + b)$

- 5** The next five expressions have a special form $(x + a)^2$ in which both linear factors are the same. Use the distributive property to find equivalent expanded forms for each given expression and look for a consistent pattern in the calculations. Remember $(x + a)^2 = (x + a)(x + a)$.

- a. $(x + 5)^2$
- b. $(x - 3)^2$
- c. $(x + 7)^2$
- d. $(x - 4)^2$
- e. $(x + a)^2$

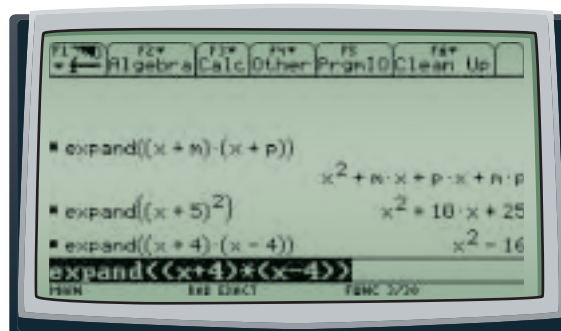
- 6** The next four expressions also have a special form in which the product can be expanded to a standard-form quadratic. Use the distributive property to find expanded forms for each expression. Then look for a pattern and an explanation of why that pattern works.

- a. $(x + 4)(x - 4)$
- b. $(x + 5)(x - 5)$
- c. $(3 - x)(3 + x)$
- d. $(x + a)(x - a)$

- 7** When algebra students see the pattern $(x + a)(x - a) = x^2 - a^2$, they are often tempted to take some other “shortcuts” that lead to errors. How would you help another student to find and correct the mistakes in these calculations?

- a. $(x + 5)(x - 3) = x^2 - 15$
- b. $(m + 7)^2 = m^2 + 49$

- 8 The next screen shows how a computer algebra system would deal with the task of expanding products of linear expressions like those in Problems 4–6. Compare these results to your own work and resolve any differences.



Summarize the Mathematics

In this investigation, you explored several of the most common ways that quadratic expressions can be written in equivalent factored and expanded forms.

- a What is the standard expanded form equivalent to the product $2x(5x + 3)$?
- b What is a factored form equivalent to $ax^2 + bx$?
- c What is the standard expanded form equivalent to the product $(x + a)(x + b)$?
- d What are the standard expanded forms equivalent to the products $(x + a)(x - a)$ and $(x + a)^2$?

Be prepared to compare your answers with those of your classmates.

✓ Check Your Understanding

Write each of the following quadratic expressions in equivalent factored or expanded form.

- a. $9x(4x - 5)$
- b. $9x^2 + 72x$
- c. $3(x^2 + 5x) - 7x$
- d. $(x + 3)(x + 7)$
- e. $(x + 2)^2$
- f. $(x + 6)(x - 6)$

In Parts g and h, find values for the missing numbers that will make the given expressions equivalent.

- g. $x^2 + 12x + \underline{\hspace{1cm}} = (x + 4)(x + \underline{\hspace{1cm}})$
- h. $x^2 + \underline{\hspace{1cm}}x - 8 = (x + 4)(x + \underline{\hspace{1cm}})$

Applications

- 1** Planners of a school fund-raising carnival considered the following factors affecting profit prospects for a rental bungee jump attraction:
- The number of customers n will depend on the price per jump x (in dollars) according to the linear function $n = 100 - x$.
 - Insurance will cost \$4 per jumper.
 - Costs include \$250 for delivery and setup and \$100 to pay a trained operator to supervise use of the jumping equipment.

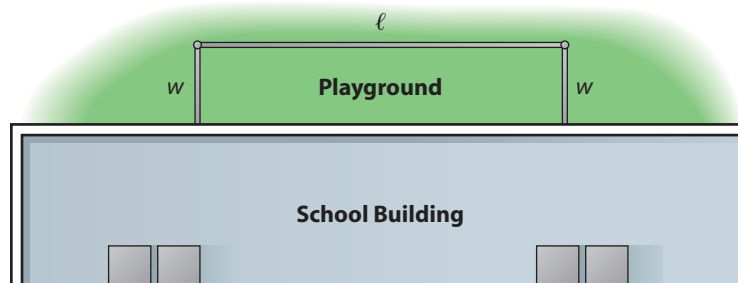


- a.** Complete a table like that begun here, showing number of customers, income, costs, and profit expected for various possible prices.

	Price per Jump (in \$)						
	0	15	30	45	60	75	90
Number of Customers							
Income (in \$)							
Insurance Cost (in \$)							
Delivery/Setup Cost (in \$)							
Operator Pay (in \$)							
Profit (in \$)							

- b.** Plot the (*price per jump*, *profit*) data. Then find a function that models the pattern relating those variables.
- c.** Write a rule showing how profit p depends on price per jump x by replacing each variable name in the following verbal rule with an expression using numbers and symbols:
- $$\text{profit} = \text{income} - \text{insurance cost} - \text{delivery/setup cost} - \text{operator pay}$$
- d.** Check to see if the expressions for profit derived in Parts b and c are equivalent and explain how you reached your conclusion.

- 2 Students in a child development class at Caledonia High School were assigned the task of designing and building a fenced playground attached to their school as shown in the following sketch. They had a total of 150 feet of fencing to work with.



- a. Complete a table like that begun here, showing how the length and area of the playground depend on choice of the width w .

Width (in feet)	10	20	30	40	50	60	70
Length (in feet)	130						
Area (in ft^2)	1,300						

- b. Plot the data relating area to width and find the function that models the pattern in that relationship.
- c. Write a function rule showing how length ℓ depends on width w and then another showing how area A depends on width.
- d. Compare the two functions from Parts b and c relating area to width and decide whether they are equivalent. Explain evidence or reasoning that supports your answer.
- e. Graph the area function and estimate the value of w that will produce the playground with largest possible area. Find the corresponding value of ℓ .

- 3 In many mountainous places, rope bridges provide the only way for people to get across fast rivers and deep valleys. A civil engineering class at a Colorado university got interested in one such rope bridge located in the mountains near their campus.

They came up with a function that they believed would give the distance in feet from the bridge to the river at any point. The function proposed was $d = 0.02x(x - 100) + 110$, where x measures horizontal distance (0 to 80 feet) from one side of the river to the other.



- a. Use the given function to calculate the distance from the bridge to the river below at points 0, 10, 20, 30, 40, 50, 60, 70, and 80 feet from one end of the bridge. Sketch a graph showing the bridge shape in relation to the mountain sides and to the river below.
- b. Estimate the low point of the bridge and its height above the water.
- c. One brave student decided to check the proposed model of the distance from the bridge to the river below. She walked across the bridge and used a range-finding device to get data relating bridge height to horizontal distance. Her data are shown in the following table.

Horizontal Distance x (in feet)	15	25	35	45	55	65	75
Distance to the River d (in feet)	85	70	65	60	60	65	75

Find a function that models the pattern in these data well.

- d. Compare the function proposed by the civil engineering students (who used only a few data points to derive their model) to that based on the range-finder data and decide whether you think the two models are equivalent or nearly so.
- e. Write the first function $d = 0.02x(x - 100) + 110$ in standard quadratic form and explain how that form either supports or undermines your decision in Part d.

- 4 Write each of the following quadratic expressions in equivalent standard form.

- a. $(3x + 4)x$
- b. $m(3m - 15)$
- c. $2p(3p - 1)$
- d. $3d(5d + 2) + 29$

- 5 Write each of these quadratic expressions in equivalent form as the product of two linear factors.

- a. $3x^2 + 9x$
- b. $2x - 5x^2$
- c. $-7d^2 - 9d$
- d. $cx + dx^2$

- 6 Write each of these quadratic expressions in two equivalent forms—one expanded and one factored—so that both are as short as possible.

- a. $2x(5 - 3x) + 4x$
- b. $-3(2s^2 + 4s) - (3s + 5)7s$
- c. $(9m + 18)m - 3m^2 - 5m$
- d. $6x(8x + 3) + 4(2x - 7) - 2x$

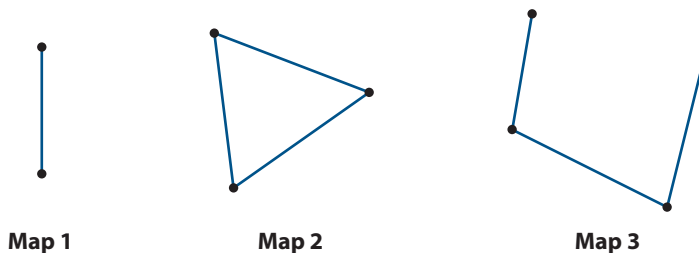
- 7 Expand each of the following products to equivalent expressions in standard quadratic form.

- a. $(x + 2)(x + 7)$
- b. $(p + 2)(p - 2)$
- c. $(x + 6)(x - 6)$
- d. $(x + 6)(x + 6)$
- e. $(R + 1)(R - 4)$
- f. $(m - 7)(7 + m)$

- 8 Expand each of the following products to equivalent expressions in standard quadratic form.
- a. $(t + 9)(t - 5)$ b. $(m + 1)^2$
 c. $(x + 9)(9 - x)$ d. $(3x + 6)(3x - 6)$

Connections

- 9 The diagrams below are vertex-edge graphs that you can think of as maps that show cities and roads connecting them. In the first two “maps,” every city can be reached from every other city by a direct road. In the third “map,” every city can be reached from every other city, but some trips would require passing through another city along the way.



- a. Sketch maps with 5, 6, and 7 cities and the smallest number of connecting roads to enable travel from any city on the map to any other city. Record the (*number of cities*, *number of roads*) data in a table like this:

Number of Cities c	2	3	4	5	6	7	8	9
Number of Roads r	1	2	3					

- i. Use the pattern of results from the sketches to find a rule for calculating the number of roads r for any number of cities c if the number of connecting roads is to be a minimum in each case.
- ii. Describe the type of function relating r and c . Explain how the rule could be justified.
- b. Next sketch maps with 4, 5, 6, and 7 cities and direct roads connecting each pair of cities. Record the (*number of cities*, *number of roads*) data in a table like this:

Number of Cities c	2	3	4	5	6	7	8
Number of Roads r	1	3					

- i. Use the pattern of results from the sketches to find a rule for calculating the number of direct roads r for any number of cities c .

ii. Study the following argument:

Each of the c cities must be connected by a road to the $c - 1$ other cities, so it looks like there must be $c(c - 1)$ direct roads. But that counting will include each road twice, so the actual number of direct roads r in a map with c cities is given by $r = \frac{c(c - 1)}{2}$.

Do the reasoning and the resulting function seem right?

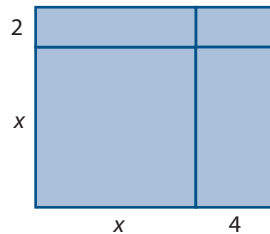
iii. Explain how you know that the results from data analysis in part i and from the reasoning approach in part ii are or are not equivalent.

- 10 Your work with the Pythagorean Theorem often involved expressions with radicals like $\sqrt{49}$ or $\sqrt{2x^2}$, where $x > 0$. Write each of the following radical expressions in equivalent form with the simplest possible whole number or expression under the radical sign.

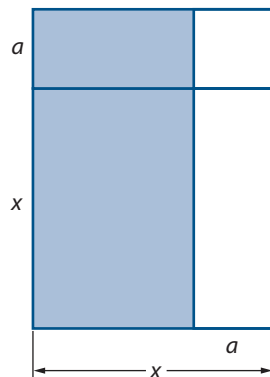
a. $\sqrt{18}$ b. $\sqrt{\frac{9}{4}}$ c. $\sqrt{2x^2}$ d. $\sqrt{\frac{3x^2}{4}}$

- 11 You can think of expanding products of linear expressions in terms of geometric models based on the area formula for rectangles.

- a. What is the expanded form of $(x + 2)(x + 4)$? How is that result shown in the following diagram? (*Hint: How can the area of the whole rectangle be calculated in two ways?*)



- b. Find the expanded form of $(x + 3)(x + 7)$ and illustrate the result with a sketch similar to that given in Part a.
- c. Make an area sketch like that in Part a to illustrate the general rule for expanding an expression in the form $(x + k)^2$.
- d. What general result does the next sketch show?



- 12** What kind of function do you think will result when two exponential functions are added or when the same two functions are multiplied? Use the examples in Parts a–d to develop conjectures from exploration of tables and graphs. Then summarize your ideas by answering Parts e and f, and use what you know about combining exponential expressions to confirm your ideas.

- a. $y_1 = 1.5^x$ and $y_2 = 2^x$
- b. $y_1 = 0.8^x$ and $y_2 = 2^x$
- c. $y_1 = 0.5^x$ and $y_2 = 0.9^x$
- d. $y_1 = 3^x$ and $y_2 = 2(3^x)$

Based on these examples (and others you might choose to test), how would you answer the questions in Parts e and f?

- e. Is the sum of two exponential functions (always, sometimes, never) an exponential function?
- f. Is the product of two exponential functions (always, sometimes, never) an exponential function?

- 13** When working with exponential growth functions it is often important to compare the value of the function at one time to the value at some future time. You can make the comparison by division or by subtraction.

- a. What properties of numbers, operations, and exponents justify each step in this reasoning that claims $3^{x+1} - 3^x = 2(3^x)$?

$$\begin{aligned} 3^{x+1} - 3^x &= (3^x)(3^1) - (3^x)(1) & (1) \\ &= 3^x(3 - 1) & (2) \\ &= 2(3^x) & (3) \end{aligned}$$

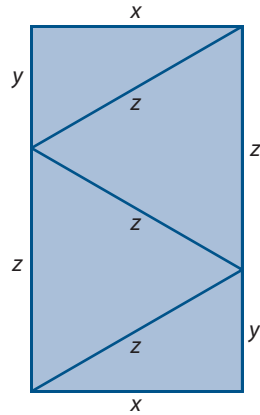
- b. Use similar reasoning to find an expression of the form $k(3^x)$ equivalent to $3^{x+2} - 3^x$.
- c. Use similar reasoning to show that for any number n , $3^{x+n} - 3^x = 3^x(3^n - 1)$.
- d. What property of exponents guarantees that for any x , $\frac{3^{x+1}}{3^x} = 3$?
- e. What property of exponents guarantees that for any x , $\frac{3^{x+n}}{3^x} = 3^n$?

Reflections

- 14** When you are working with a quadratic function with rule like $y = 5x^2 + 15x$, what kinds of questions would be most easily answered using the rule in that standard form and what kinds of questions are easier to answer when the rule uses the equivalent expression $y = 5x(x + 3)$?
- 15** The claims below show some of the most common errors that people make when attempting to write quadratic expressions in equivalent forms by expanding, factoring, and rearranging terms. Spot the error(s) in each claim and tell how you would help the person who made the error correct his or her understanding.

- a. Claim: $5x(4 + 3x)$ is equivalent to $23x$.
- b. Claim: $7x - 5(2x + 4)$ is equivalent to $-3x + 20$.
- c. Claim: $5x^2 + 50x$ is equivalent to $5x(x + 50)$.
- d. Claim: $5x + 7x^2$ is equivalent to $12x^3$.

- 16** You know at least four different strategies for checking to see if two algebraic expressions are equivalent or not—comparing tables or graphs of (x, y) values, comparing the reasoning that led from the problem conditions to the expressions, or using algebraic reasoning based on number system properties like the distributive property.
- a. What do you see as the advantages and disadvantages of each strategy?
 - b. How do you decide which strategy to use in a given situation?
 - c. Which strategy gives you most confidence in your judgment about whether the given expressions are or are not equivalent?
- 17** A rectangle is divided into 4 regions by equal-length segments as shown.



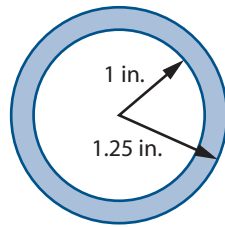
- a. Bushra says the area of the rectangle is $x(y + z)$. Is she correct? If so, explain the reasoning she likely used.
- b. Kareem says that the area of the rectangle is $\frac{1}{2}zx + \frac{1}{2}zx + \frac{1}{2}xy + \frac{1}{2}xy$. Is he correct? If so, explain his possible reasoning.
- c. Fatmeh says that the area is $2x + 2y + 2z$. Is she correct? If so, explain her reasoning.

Extensions

- 18** Suppose that the conditions for operation of a rented bungee jump at the school carnival are as follows:
- The number of customers n will depend on the price per jump x (in dollars) according to the linear function $n = 80 - 0.75x$.
 - Insurance will cost \$500 plus \$2 per jumper.
 - Costs include \$250 for delivery and setup and \$3 per jumper to pay a trained operator to supervise use of the jumping equipment.
- a. Write two rules for calculating projected profit for this attraction as a function of price per jump x . Write one rule in a form that shows how each income and cost factor contributes and another that is more efficient for calculation. Explain how you are sure that the two rules are equivalent.
 - b. Use one of the profit rules from Part a to estimate the price that will yield maximum profit and to find what that profit is.
 - c. Use one of the profit rules from Part a to estimate price(s) that will assure at least some profit (not a loss) for the attraction.
- 19** The sum of two linear expressions is always a linear expression, but the product of two linear expressions is *not always* a quadratic expression.
- a. Use what you know about rearranging and combining terms in a linear expression to prove that $(ax + b) + (cx + d)$ is always a linear expression.
 - b. Find examples of linear expressions $ax + b$ and $cx + d$ for which the product $(ax + b)(cx + d)$ is not a quadratic expression.
 - c. Under what conditions will the product of two linear expressions *not be* a quadratic expression?
- 20** Consider the following mathematical question.
- If a 5-ft tall person stands in one spot on the equator of Earth for 24 hours, how much farther will that person's head travel than his or her feet as Earth rotates about its axis?*
- a. Before doing any calculations, which of the following would you guess as an answer to the question?

1 foot	30 feet	300 feet	3,000 feet
1 mile	30 miles	300 miles	3,000 miles
 - b. Use the facts that the radius of Earth is about 4,000 miles at the equator and there are 5,280 feet in one mile to find the answer.
 - c. Use algebraic reasoning to answer this more general question:
If a tower that is k feet tall stands upright on the equator of a sphere of radius r feet, how much farther will the top of the tower travel than the base of the tower as the sphere makes one complete revolution about its axis?
 Show that your answer works for the case of the person standing on Earth's equator.

- d. The next sketch shows a disc inside a shaded ring. If the disc has radius 1 inch and the ring adds 0.25 inches to the radius of the figure, what is the area of the shaded ring?

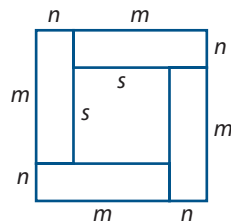


- e. If a disc of radius r is inside a shaded ring that adds k to the radius, what rule gives the area of the shaded ring? Express that rule using the simplest possible expression involving r and k .
- f. Natasha thought you might be able to calculate the area of the shaded ring by thinking about unwrapping the ring from around the disc. She said it would be pretty close to a rectangle, so its area could be estimated by multiplying the circumference of the ring by the width of the ring. She thought of three possible calculations to estimate this area.

$$2\pi rk \qquad 2\pi(r + k)k \qquad 2\pi\left(r + \frac{k}{2}\right)k$$

- i. What thinking would have led Natasha to each of these expressions for area of the shaded ring?
- ii. Which, if any, of the expressions gives a correct way of estimating the area of the shaded ring?

- 21** Study the geometric design in the figure below—a square with sides of length s surrounded by four congruent rectangles.



- a. Express s in terms of m and n .
- b. Express the area of the large square in two equivalent ways—each a function of m and n .
- c. Equate the two expressions for area you found in Part b. Then explain how this equation implies that $(m + n)^2 \geq 4mn$ for any positive numbers m and n .
- d. Use the result in Part c to explain why $\frac{m + n}{2} \geq \sqrt{mn}$ for any positive numbers m and n . This is known as the *arithmetic mean-geometric mean inequality*.

- 22** Expand each expression and look for a pattern to shortcut the calculations.
- a. $(3x + 5)(2x + 1)$ b. $(5x - 3)(x + 4)$
 c. $(-2x + 7)(4x - 3)$ d. $(7x - 4)(x + 2)$
 e. $(ax + b)(cx + d)$
- 23** Find equivalent expanded forms for each given expression and look for a pattern to shortcut the calculations.
- a. $(3x + 5)^2$ b. $(5x - 3)^2$
 c. $(-2x + 7)^2$ d. $(7x - 4)^2$
 e. $(ax + b)^2$
- 24** Find equivalent expanded forms for each given expression and look for a pattern to shortcut the calculations.
- a. $(3x + 5)(3x - 5)$ b. $(2x - 3)(2x + 3)$
 c. $(-2x + 7)(-2x - 7)$ d. $(8 - 4x)(8 + 4x)$
 e. $(ax + b)(ax - b)$
- 25** Consider all quadratic functions with rules of the form $y = ax^2 + bx$.
- a. How can any expression $ax^2 + bx$ be written as a product of linear factors?
- b. Why does the factored form in Part a imply that $ax^2 + bx = 0$ when $x = 0$ and when $x = \frac{-b}{a}$?
- c. How does the information from Part b imply that the maximum or minimum point on the graph of any function $y = ax^2 + bx$ occurs where $x = \frac{-b}{2a}$?

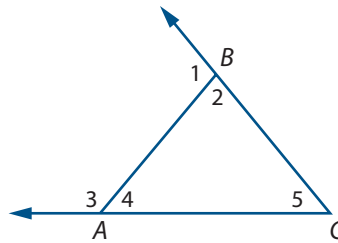
Review

- 26** Solve these linear equations.
- a. $3.5x + 5 = 26$ b. $7(2x - 9) = 42$
 c. $\frac{5x}{2} + 12 = 99$ d. $3.5x - 8 = 12 + 5x$
 e. $7x + 4 = 4(x + 7)$ f. $3(x + 2) = 1.5(4 + 2x)$
- 27** Amy's last 10 scores on 50-point quizzes are 30, 32, 34, 34, 35, 35, 35, 36, 40, 42.
- a. She says she can find her mean score by adding 3.0, 3.2, 3.4, 3.4, 3.5, 3.5, 3.5, 3.6, 4.0, and 4.2. Her friend Bart says she has to add and divide the total by 10, $\frac{30 + 32 + 34 + 34 + 35 + 35 + 35 + 36 + 40 + 42}{10}$. Who is correct? Explain why.

- b. Amy is hoping for a 44 on the next 50-point quiz. If so, she says she will add 4.4 to her current mean to get the new mean. Is she correct?
- c. Amy says that if she gets a 44 on the next quiz, her median score will increase. Bart disagrees. Who is correct? Explain.

- 28 In the diagram at the right, $\overline{AB} \cong \overline{BC}$ and $m\angle 1 = 100^\circ$. Use geometric reasoning to determine:

- a. $m\angle 2$ b. $m\angle 4 + m\angle 5$
c. $m\angle 4$ d. $m\angle 3$



- 29 Try using various values of a , b , and c to help you determine if each pair of expressions are equivalent. If you think they are not equivalent, show this with an example using values for a , b , and c . If you think they are equivalent, use algebraic reasoning to transform one into the other.

- a. $(a + b)^2$ and $a^2 + b^2$
b. $a(b - c)$ and $ab - ac$
c. $\sqrt{a^2 + b^2}$ and $a + b$
d. $\frac{\sqrt{9a^2}}{3}$ and a ($a \geq 0$)
e. $(ab)^c$ and $a^c b^c$

- 30 Alex takes two servings of fruit with him every day to school. Suppose that there are always apples, oranges, grapes, and watermelon in his house.

- a. List all the possibilities for his daily fruit if he wants two different types of fruit.
- b. How much longer would your list be if he didn't care if the types of fruit were different? Explain your reasoning.

- 31 Draw a triangle that has a perimeter of 12 cm.

- a. How long are the sides of your triangle?
- b. Suppose that another student in your class drew a triangle with the same side lengths as yours. Will the two triangles be guaranteed to be congruent? Explain your reasoning.
- c. Heather wants to draw a right triangle that has perimeter 12 cm. Give possible lengths of the sides that she could use. Explain how you know they will give her a right triangle.
- d. Christine indicates that the triangle she drew has sides of length 7 cm, 3 cm, and 2 cm. Is that possible? Explain why or why not.

LESSON 3

Solving Quadratic Equations

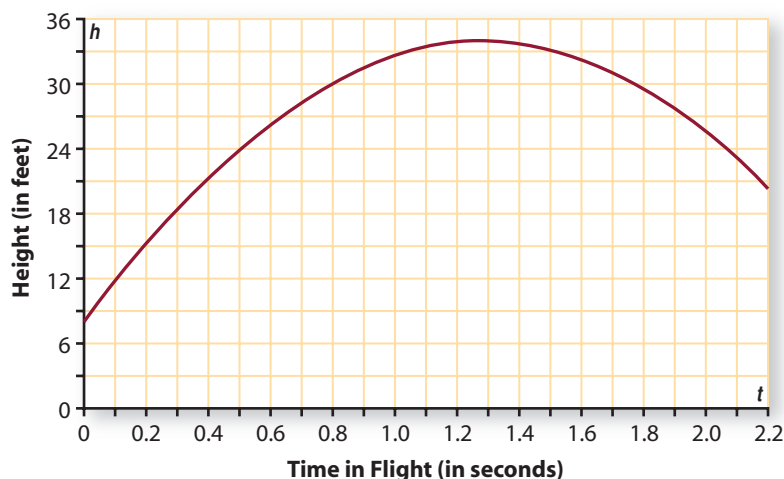
Many key questions about quadratic functions require solving equations. For example, you used the function $h = 8 + 40.8t - 16t^2$ to answer questions about the flight of a long basketball shot.

To find the time when the shot would reach the 10-foot height of the basket, you solved the equation $8 + 40.8t - 16t^2 = 10$.

To find the time when an “air ball” would hit the floor, you solved the equation $8 + 40.8t - 16t^2 = 0$.

The values of t that satisfy the equations are called the **solutions** of the equations.

In each problem, you could get good estimates of the required solutions by searching in tables of values or by tracing coordinates of points on the graphs of the height function.



t	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
h	8	15.2	21.8	26.7	30.4	32.8	33.9	33.8	32.3	29.6	25.6	20.3

Think About This Situation

Based on what you know about the tables and graphs of quadratic functions:

- a** How many solutions would you expect for each of these equations, and what strategies would you use to find those solutions?
- i. $8 + 40.8t - 16t^2 = 10$
 - ii. $8 + 40.8t - 16t^2 = 0$
 - iii. $8 + 40.8t - 16t^2 = 40$
 - iv. $8 + 40.8t - 16t^2 = 34$
- b** How many solutions would you expect for other quadratic equations in the form $ax^2 + bx + c = d$?

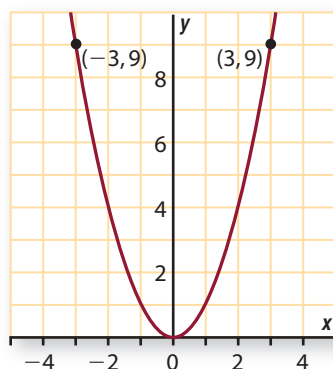
In this lesson, you will explore methods for finding exact solutions of quadratic equations.

Investigation 1 Solving $ax^2 + c = d$ and $ax^2 + bx = 0$

Calculator or computer tables, graphs, and symbol manipulation tools are helpful in finding approximate or exact solutions for quadratic equations and inequalities. But there are times when it is easier to use algebraic reasoning alone. As you work on the problems of this investigation, look for answers to this question:

What are some effective methods for solving quadratic equations algebraically?

- 1** Some quadratic equations can be solved by use of the fact that for any positive number n , the equation $x^2 = n$ is satisfied by two numbers: \sqrt{n} and $-\sqrt{n}$.



Use this principle and what you know about solving linear equations to solve the following quadratic equations. In each case, check your reasoning by substituting the proposed solution values for x in the original equation.

a. $x^2 = 25$

b. $x^2 = 12$

c. $5x^2 = 60$

d. $5x^2 + 8 = 8$

e. $5x^2 + 15 = 60$

f. $5x^2 + 75 = 60$

g. $-5x^2 + 75 = 60$

h. $x^2 = -16$

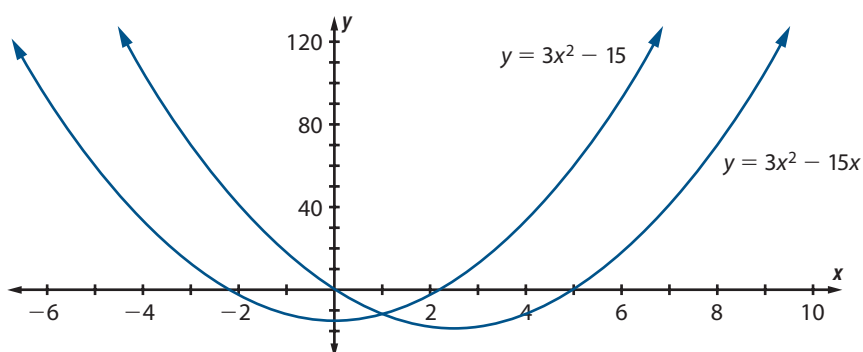
- 2** Use the methods you developed in reasoning to solutions for equations in Problem 1 to answer these questions about flight of a platform diver.

- If the diver jumps off a 50-foot platform, what rule gives her or his distance fallen d (in feet) as a function of time t (in seconds)?
- Write and solve an equation to find the time required for the diver to fall 20 feet.
- What function gives the height h (in feet) of the diver at any time t (in seconds) after she or he jumps from the platform?
- Write and solve an equation to find the time when the diver hits the water.

- 3** If a soccer player kicks the ball from a spot on the ground with initial upward velocity of 24 feet per second, the height of the ball h (in feet) at any time t seconds after the kick will be approximated by the quadratic function $h = 24t - 16t^2$. Finding the time when the ball hits the ground again requires solving the equation $24t - 16t^2 = 0$.

- Check the reasoning in this proposed solution of the equation.
 - The expression $24t - 16t^2$ is equivalent to $8t(3 - 2t)$. Why?
 - The expression $8t(3 - 2t)$ will equal 0 when $t = 0$ and when $3 - 2t = 0$. Why?
 - So, the solutions of the equation $24t - 16t^2 = 0$ will be 0 and 1.5. Why?
- Adapt the reasoning in Part a to solve these quadratic equations.
 - $0 = x^2 + 4x$
 - $0 = 3x^2 + 10x$
 - $0 = x^2 - 4x$
 - $-x^2 - 5x = 0$
 - $-2x^2 - 6x = 0$
 - $x^2 + 5x = 6$

- 4** Solving quadratic equations like $3x^2 - 15 = 0$ and $3x^2 - 15x = 0$ locates x -intercepts on the graphs of the quadratic functions $y = 3x^2 - 15$ and $y = 3x^2 - 15x$.



- a. Using the graphs above, explain how the symmetry of these parabolas can be used to relate the location of the minimum (or maximum) point on the graph of a quadratic function to the x -intercepts.
- b. Use the results of your work in Problem 3 to find coordinates of the maximum or minimum points on the graphs of these quadratic functions.
 - i. $y = x^2 + 4x$
 - ii. $y = 3x^2 + 10x$
 - iii. $y = x^2 - 4x$
 - iv. $y = -x^2 - 5x$
 - v. $y = -2x^2 - 6x$
 - vi. $y = ax^2 + bx$

- 5** Use what you know about solving quadratic equations and the graphs of quadratic functions to answer these questions.

- a. What choices of values for a and d will give equations in the form $ax^2 = d$ that have two solutions? Only one solution? No solutions? Explain the reasoning behind your answers and illustrate that reasoning with sketches of graphs for the related function $y = ax^2$.
- b. What choices of values for a , c , and d will give equations in the form $ax^2 + c = d$ that have two solutions? Only one solution? No solutions? Explain the reasoning behind your answers and illustrate that reasoning with sketches of graphs for the related function $y = ax^2 + c$.
- c. Why must every equation in the form $ax^2 + bx = 0$ (neither a nor b zero) have exactly two solutions? Explain the reasoning behind your answer and illustrate that reasoning with sketches of graphs for the related function $y = ax^2 + bx$.

Summarize the Mathematics

In this investigation, you learned how to find exact solutions for some forms of quadratic equations by algebraic reasoning.

- a** Describe a process that uses rules of algebra to find solutions for any quadratic equation in the form $ax^2 + c = d$.
- b** Describe a process that uses rules of algebra to find solutions for any quadratic equation in the form $ax^2 + bx = 0$.
- c** What are the possible numbers of solutions for equations in the form $ax^2 = d$? For equations in the form $ax^2 + c = d$? For equations in the form $ax^2 + bx = 0$?
- d** How can graphs of quadratic functions in the form $y = ax^2$, $y = ax^2 + c$, and $y = ax^2 + bx$ be used to illustrate your answers to Part c?
- e** How can you locate the maximum or minimum point on the graph of a quadratic function with rule in the form:

i. $y = ax^2$

ii. $y = ax^2 + c$

iii. $y = ax^2 + bx$

Be prepared to share your ideas and reasoning strategies with the class.

Check Your Understanding

Quadratic equations, like linear equations, can often be solved more easily by algebraic reasoning than by estimation using a graph or table of values for the related quadratic function.

- a.** Solve each of these quadratic equations algebraically.
 - i.** $3x^2 = 36$
 - ii.** $-7x^2 = 28$
 - iii.** $5x^2 + 23 = 83$
 - iv.** $-2x^2 + 4 = 4$
 - v.** $7x^2 - 12x = 0$
 - vi.** $x^2 + 2x = 0$
- b.** Use algebraic methods to find coordinates of the maximum and minimum points on graphs of these quadratic functions.
 - i.** $y = -5x^2 - 2$
 - ii.** $y = 5x^2 + 3$
 - iii.** $y = x^2 - 8x$
 - iv.** $y = -x^2 + 2x$

Investigation 2 The Quadratic Formula

Many problems that require solving quadratic equations involve *trinomial* expressions like $15 + 90t - 16t^2$ that are not easily expressed in equivalent factored forms. So, solving equations like

$$15 + 90t - 16t^2 = 100$$

(When is a flying pumpkin 100 feet above the ground?)

is not as easy as solving equations like those in Investigation 1.

Fortunately, there is a **quadratic formula** that shows how to find all solutions of any quadratic equation in the form $ax^2 + bx + c = 0$. For any such equation, the solutions are

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

In Course 3 of *Core-Plus Mathematics*, you will prove that the quadratic formula gives the solutions to any quadratic equation. For now, to use the quadratic formula in any particular case, all you have to do is

- be sure that the quadratic expression is set equal to 0 as is prescribed by the formula;
- identify the values of a , b , and c ; and
- substitute those values where they occur in the formula.

As you work on the problems in this investigation, make notes of answers to these questions:

What calculations in the quadratic formula give information on the number of solutions of the related quadratic equation?

What calculations provide information on the x -intercepts and maximum or minimum point of the graph of the related quadratic function?

1 Solve each quadratic equation by following the procedure for applying the quadratic formula.

- Give the values of a , b , and c that must be used to solve the equations by use of the quadratic formula.
- Evaluate $\frac{-b}{2a}$ and $\frac{\sqrt{b^2 - 4ac}}{2a}$.
- Evaluate $x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$.
- Check that the solutions produced by the formula actually satisfy the equation.

a. $2x^2 - 2x - 12 = 0$

b. $15 + 90t - 16t^2 = 100$

2 Test your understanding and skill with the quadratic formula by using it to find solutions for the following equations. In each case, check your work by substituting proposed solutions in the original equation and by sketching a graph of the related quadratic function to show how the solutions appear as points on the graph.

a. $x^2 - 7x + 10 = 0$

b. $x^2 - x - 8 = 0$

c. $-x^2 - 3x + 10 = 0$

d. $2x^2 - 12x + 18 = 0$

e. $13 - 6x + x^2 = 0$

f. $-x^2 - 4x - 2 = 2$

- 3** The formula for calculating solutions of quadratic equations is a complex set of directions. You can begin to make sense of the formula, by connecting it to patterns in the graphs of quadratic functions.

Consider the related quadratic functions:

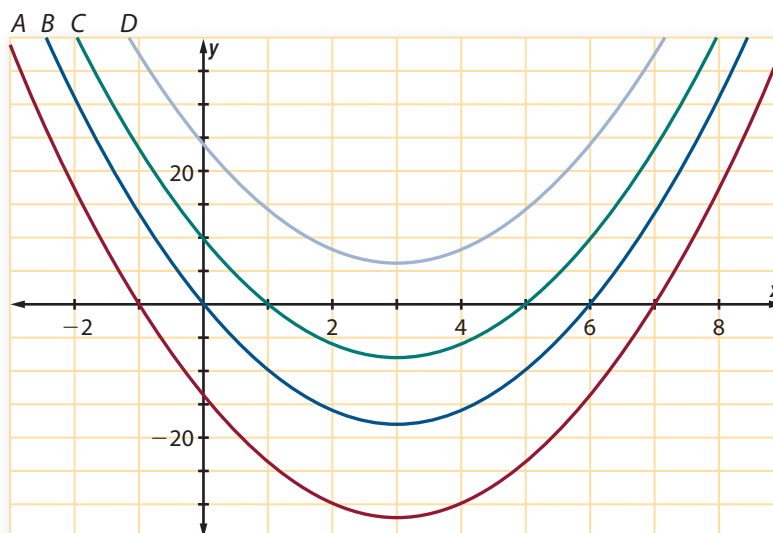
$$y = 2x^2 - 12x$$

$$y = 2x^2 - 12x + 10$$

$$y = 2x^2 - 12x - 14$$

$$y = 2x^2 - 12x + 24$$

The graphs of these functions are shown in the following diagram.



- Match each function with its graph.
- What are the x -coordinates of the minimum points on each graph? How (if at all) are those x -coordinates related to the x -intercepts of the graphs?
- What are the coordinates of the x -intercepts on the graph of a quadratic function with rule $y = ax^2 + bx$? What is the x -coordinate of the minimum or maximum point for such a graph?
- How will the x -coordinate of the maximum or minimum point of $y = ax^2 + bx + c$ be related to that of $y = ax^2 + bx$?

- 4** Now look back at the quadratic formula and think about how the results of Problem 3 help to explain the connections between these parts of the formula and the graph of $y = ax^2 + bx + c$.

a. $\frac{-b}{2a}$

b. $\frac{\sqrt{b^2 - 4ac}}{2a}$

c. $\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$

d. $\frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$

Summarize the Mathematics

This investigation focused on a method for finding exact solutions for any quadratic equation.

- a** What are the key steps in using the quadratic formula to solve a quadratic equation?
- b** How can you tell from the calculation $\sqrt{b^2 - 4ac}$ that is part of the quadratic formula when a given quadratic equation has 2, 1, or 0 solutions? Give specific examples that illustrate the three possibilities.
- c** How are the locations of the maximum or minimum point and x-intercepts of a quadratic function graph shown by calculations in the quadratic formula?
- d** How do you choose among solution strategies—estimation using a table or graph, using arithmetic operations and square roots, factoring, or the quadratic formula—for the different quadratic equations that arise in solving problems?

Be prepared to share your thinking with the entire class.

Check Your Understanding

Use what you have learned about the quadratic formula to complete the following tasks.

- a.** Solve each of these equations. Show or explain the steps in each solution process.
 - i.** $x^2 - 6x + 8 = 0$
 - ii.** $3x^2 - 8 = -2$
 - iii.** $x^2 - 2x + 8 = 2$
 - iv.** $-7x + 8 + x^2 = 0$
 - v.** $15 = 7 + 4x^2$
 - vi.** $x^2 = 36$
- b.** Suppose that a group of students made these statements in a summary of what they had learned about quadratic functions and equations. With which would you agree? For those that you don't believe to be true, what example or argument would you offer to correct the proposer's thinking?
 - i.** "Every quadratic equation has two solutions."
 - ii.** "The quadratic formula cannot be applied to an equation like $-7x + 8 + x^2 = 0$."
 - iii.** "To use the quadratic formula to solve $-7x + 8 + x^2 = 0$, you let $a = -7$, $b = 8$, and $c = 1$."
 - iv.** "To use the quadratic formula to solve $x^2 - 6x + 8 = 0$, you let $a = 0$, $b = 6$, and $c = 8$."
 - v.** "To use the quadratic formula to solve $3x^2 + 5x - 7 = 8$, you let $a = 3$, $b = 5$, and $c = -7$."

On Your Own

Applications

- 1 Solve each of these quadratic equations by using only arithmetic operations and square roots. Show the steps of your solution process and a check of the solutions.
 - a. $x^2 = 20$
 - b. $s^2 + 9 = 25$
 - c. $x^2 - 11 = -4$
 - d. $3m^2 + 9 = 5$
 - e. $-2x^2 + 24 = 2$
 - f. $29 - 3n^2 = 5$
- 2 An engineer designed a suspension bridge so that the main cables would lie along parabolas defined by the function $h = 0.04x^2 + 15$ where h is the distance from the cable to the bridge surface at a point x feet from the *center* of the bridge. The bridge is to be 50 feet long.
 - a. Sketch a graph of the function for $-25 \leq x \leq 25$ and indicate on that sketch the height of each tower and the shortest distance from the cable to the bridge surface.
 - b. Write and solve an equation to answer the question, “At what location(s) on the bridge surface is the suspension cable exactly 20 feet above the surface?”
- 3 The exterior of the St. Louis Abbey Church in Missouri shows a collection of parabolic faces.



Suppose that an arch in the lower ring of the parabolic roof line is defined by the function $y = -0.075x^2 + 30$, with y giving the distance in feet from the roof to the ground at any point x feet from the line of symmetry for the arch.

- a. What is the maximum height of the arch and how can you find that without any calculation?

- b. Write and solve an equation that determines the distance from the line of symmetry to the points where the arch would meet the ground.
- c. Write and solve an equation that determines the points where the roof is exactly 15 feet above the ground.

4 Solve each of these quadratic equations algebraically. Show your work.

a. $5x^2 + 60x = 0$

b. $-5x^2 + 23x = 0$

c. $-12x + 7x^2 = 0$

d. $2x - x^2 = 0$

5 Without graphing, find coordinates of the maximum and minimum points on graphs of these quadratic functions.

a. $y = 5x^2 + 60x$

b. $y = -5x^2 + 23x$

c. $y = -12x + 7x^2$

d. $y = 2x - x^2$

6 In football, when a field goal attempt is kicked, it leaves the ground on a path for which the height of the ball h in feet at any time t seconds later might be given by a function like $h = 45t - 16t^2$.

- a. Write and solve an equation that tells time(s) when the ball hits the ground at the end of its flight.
- b. Write and solve an equation that tells time(s) when the ball is at the height of the end zone crossbar (10 feet above the ground).
- c. Find the maximum height of the kick and when it occurs.

7 Using the quadratic function $y = x^2 - 3x + 2$, choose values of y to write equations that have the prescribed number of solutions. In each case, show on a graph of the function how the condition is satisfied.

- a. Two solutions
- b. One solution
- c. No solutions

8 Test your understanding and skill with the quadratic formula by using it to find solutions for the following equations. In each case, check your work by substituting proposed solutions in the original equation and by sketching a graph of the related quadratic function.

a. $x^2 + 4x + 5 = 0$

b. $x^2 - 7x + 8 = -2$

c. $x^2 - x - 8 = 4$

d. $5x + x^2 - 3 = 0$

e. $3x^2 - 18x + 27 = 0$

f. $5x^2 - x = 8$



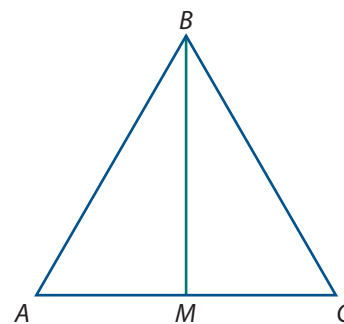
Connections

- 9** The Pythagorean Theorem says that in any right triangle with legs of length a and b and hypotenuse of length c , $c^2 = a^2 + b^2$. Write and solve quadratic equations that provide answers for these questions:

- What is the length of the hypotenuse of a right triangle with legs of length 5 and 7?
- If a right triangle has one leg of length 9 and the hypotenuse of length 15, what is the length of the other leg?
- If a right triangle has hypotenuse of length 20 and one leg of length 10, what is the length of the other leg?
- What are the lengths of the sides of a square that has diagonal length 15?

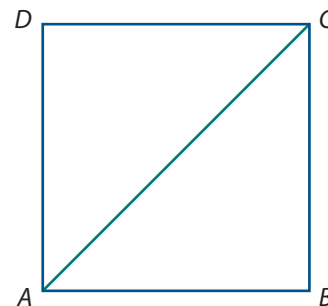
- 10** Shown at the right is an equilateral triangle $\triangle ABC$. M is the midpoint of \overline{AC} .

- What are the measures of the angles in $\triangle ABM$?
- If side \overline{AB} has length 8, what is the length of side \overline{AM} ? What is the length of side \overline{MB} ?
- If side \overline{AB} has length x , what are the lengths of sides \overline{AM} and \overline{MB} in terms of x ?



- 11** At the right is square $ABCD$ with one diagonal drawn in.

- If side \overline{AB} has length 6, what is the length of the diagonal \overline{AC} ?
- If side \overline{AB} has length x , show that the length of the diagonal \overline{AC} is $x\sqrt{2}$.



- 12** Consider all linear equations of the type $2x + 1 = k$.

- What are the greatest and the least numbers of solutions for such equations?
- How can your answer to Part a be supported with a sketch of the graph for the function $y = 2x + 1$?

- 13** Consider all equations of the type $5(2^x) = k$. How many solutions can there be for such equations? Give equations with specific values of k that illustrate the possibilities. How can you support your answer with a graph of the function $y = 5(2^x)$?

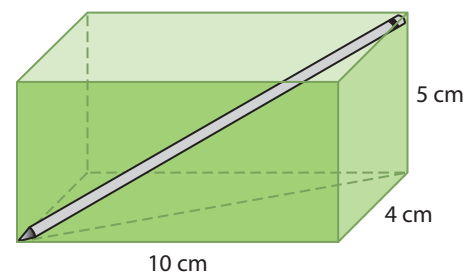
- 14** The quadratic formula gives a rule for finding all solutions of equations in the form $ax^2 + bx + c = 0$. Now consider all linear equations of the type $ax + b = k$ (where $a \neq 0$). Write a rule that shows how to calculate all solutions of such linear equations, using arithmetic operations involving the values of a , b , and k .

Reflections

- 15** What are the possible numbers of solutions for a quadratic equation? How would you use the graph of a quadratic function to explain those possibilities?
- 16** Mathematicians call " $x^2 + 5x + 6$ " an expression and " $x^2 + 5x + 6 = 0$ " an equation. How would you explain the difference between *expressions* and *equations*?
- 17** How are the possible numbers of solutions for a quadratic equation seen by examining the quadratic formula?
- 18** You now know three different ways to solve quadratic equations using algebraic methods: by reducing the problem to solving $x^2 = n$, by reducing the problem to solving $x(x + m) = 0$, and by using the quadratic formula.
- What do you see as the advantages and disadvantages of each strategy?
 - Give an example of a quadratic equation that you would prefer to solve by factoring.
 - Give an example of a quadratic equation that you would prefer to solve by use of the quadratic formula.

Extensions

- 19** Find all solutions of the following equations that involve products of three or four linear terms. Then sketch graphs of the functions involved in these equations for $-5 \leq x \leq 5$ and explain how the solutions to the equations are shown on the graphs. Compare the pattern relating graphs and equations in these examples to the patterns you met in dealing with quadratic functions.
- $x(x - 4)(x + 2) = 0$
 - $(2x + 3)(x - 1)(x + 3) = 0$
 - $x(2x + 3)(x - 1)(x + 3) = 0$
 - $x(x - 4)(x + 2) = -9$
- 20** Shown at the right are the dimensions for a multi-use gift box.
- Will a pen 11 cm long fit in the bottom of the box? Explain why or why not.
 - Find the length of the longest pencil that will fit inside the box. Illustrate and explain how you found your answer.





21 What values of x satisfy these quadratic inequalities?

a. $9 - x^2 < 0$

b. $9 - x^2 > 5$

c. $x^2 - 5x < 0$

d. $x^2 - 5x > 0$

22 In Investigation 1, you considered a quadratic model for a soccer ball kicked from a spot on the ground using the model $h = 24t - 16t^2$, where height was measured in feet and time in seconds.

a. What question can be answered by solving the inequality

$24t - 16t^2 > 8$?

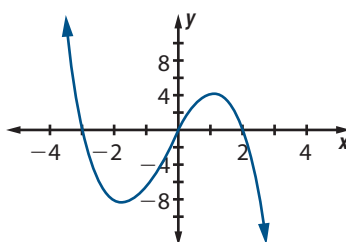
b. Solve the inequality and answer the question you posed.

c. Write an inequality that can be used to answer the question, "At what times in its flight is the ball within 8 feet of the ground?"

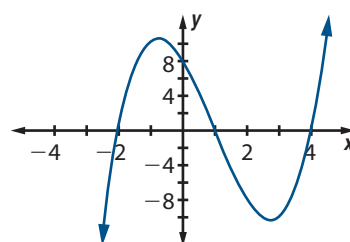
d. Solve the inequality you wrote in Part c.

23 Find rules for the functions with the graphs given below. Be prepared to explain how you developed the rules in each case.

a.



b.



24 What results will you expect from entering these commands in a calculator or computer algebra system?

a. **Solve**($a \cdot x^2 + c = d, x$)

b. **Solve**($a \cdot x^2 + b \cdot x = 0, x$)

c. **Solve**($a \cdot x^2 + b \cdot x + c = 0, x$)

Execute the commands and resolve any differences between what you expected and what occurred.

25 If you had a job that required solving many quadratic equations in the form $ax^2 + bx + c = 0$, it might be helpful to write a spreadsheet program that would do the calculations after you entered the values of a , b , and c . The table that follows was produced with one such spreadsheet and the equation $3x^2 - 6x - 24$.

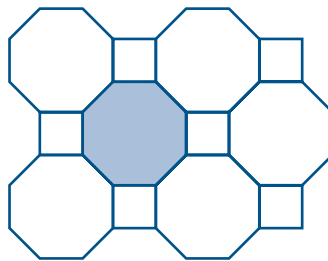
Quadratic Equations.xls				
	A	B	C	D
1	1	3	$a =$	3
2	Solution 1 =	-2	$b =$	-6
3	Solution 2 =	4	$c =$	-24
4				

a. What formulas would you expect to find in cells A1 and B1?

- 26** The graphs of $y_1 = x^2 + x - 8$ and $y_2 = 2x + 4$ are shown at the right.
- Estimate the coordinates of points where the two graphs intersect.
 - Solve the equation $x^2 + x - 8 = 2x + 4$ algebraically and compare the results to your estimate in Part a.
 - What are the possible numbers of solutions for the equation $\text{quadratic expression} = \text{linear expression}$? Use sketches of function graphs to illustrate.

Review

- 28** Examine this portion of a semiregular tessellation made with regular octagons and squares.



- 29** Given that two vertex-edge graphs both have 8 edges and 6 vertices, which of the following statements is true? Give examples to support your answers.
- a.** The graphs must be identical.
 - b.** Both graphs must have Euler circuits.
 - c.** Both graphs must have Euler paths.

30

You know that the general form of a linear function is $y = mx + b$, and that the general form of an exponential function is $y = a(b^x)$.

- a. What is the equation of a line with slope $\frac{2}{3}$ and y -intercept at $(0, 3)$?

Can you always write the equation of a line if you know its slope and y -intercept?

- b. What is the equation of a line with slope $\frac{2}{3}$ and x -intercept at $(3, 0)$?

Can you always write the equation of a line if you know its slope and x -intercept?

- c. What is the rule for an exponential function that has a graph with y -intercept at $(0, 5)$ and growth factor 2?

Can you always write the rule for an exponential function if you know the y -intercept of its graph and the growth factor?

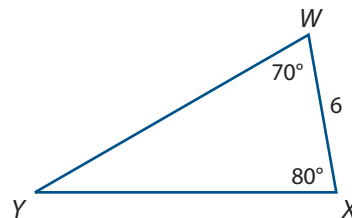
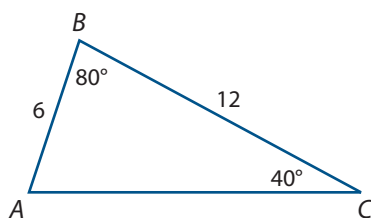
- d. Is there an exponential function matching a graph that has x -intercept at $(3, 0)$ and a decay factor of 0.5?

Can you always find an exponential function when given an x -intercept and a decay factor?

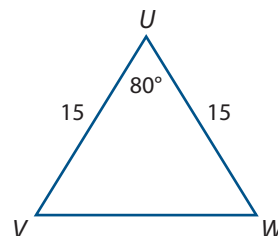
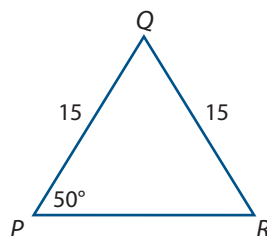
31

Determine if the triangles in each pair are congruent. If they are congruent, indicate how you know the triangles are congruent and write the congruence relation. If they are not congruent, explain how you know.

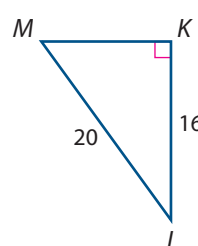
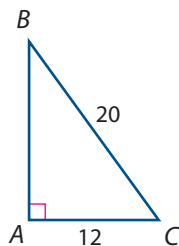
a.



b.



c.



- 32** Write results of these calculations in standard scientific notation $N \times 10^x$, $1 \leq N < 10$.
- a.** $(5 \times 10^8)(4 \times 10^{11})$
 - b.** $(5 \times 10^8)(4 \times 10^{-3})$
 - c.** $(5 \times 10^8) \div (4 \times 10^5)$
- 33** Find the value of each absolute value expression.
- a.** $|-5|$
 - b.** $-|6|$
 - c.** $|7 - 4|$
 - d.** $|4 - 7|$
 - e.** $|3(-2) + 5|$
 - f.** $|-3 - 10|$
- 34** In kite $ABCD$, $m\angle BAD = 78^\circ$ and $m\angle BCD = 40^\circ$. Calculate the measures of $m\angle ABC$ and $m\angle ADC$.

LESSON 4

Looking Back

In your work on problems and explorations of this unit, you studied patterns of change in variables that can be modeled well by quadratic functions.

The height of a basketball shot increases to a maximum and then falls toward the basket or the floor, as a function of time after the shot is taken.

The projected income from a business venture rises to a maximum and then falls, as a function of price charged for the business product.

The main cables of a suspension bridge dip from each tower to a point where the bridge surface is closest to the water below.

The stopping distance for a car increases in a quadratic pattern as its speed increases.

The functions that model these patterns of change can all be expressed with rules like

$$y = ax^2 + bx + c.$$

Since quadratic expressions play such an important role in reasoning about relations among quantitative variables, you explored a variety of strategies for combining and expanding those expressions to produce useful equivalent forms. You also learned how to solve quadratic equations by algebraic reasoning and use of the quadratic formula.

As a result of your work on Lessons 1–3, you should be better able to recognize situations in which variables are related by quadratic functions, to use data tables and graphs to display patterns in those relationships, to use symbolic expressions to describe and reason about the patterns, and to use graphing calculators and computer algebra systems to answer questions that involve quadratic functions.

In future units of *Core-Plus Mathematics*, you will extend your understanding and skill in use of quadratic functions and expressions to solve problems. The tasks in this final lesson ask you to put your current knowledge to work in solving several new problems.

Mystic Mountain Mike and Tanya grew up in the mountains of Idaho, but they are now mathematics teachers in a large eastern city. They have a dream of going into business with a restaurant and entertainment complex they plan to call Mystic Mountain.

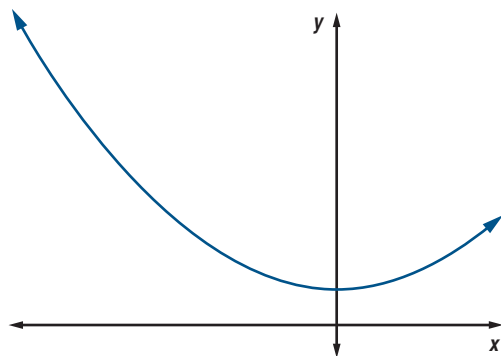


Not surprisingly, several of Mike and Tanya's ideas for Mystic Mountain involve quadratic functions and their graphs.

1 The Entry Bridge Their first idea is to have customers enter Mystic Mountain by walking across a rope bridge suspended over an indoor river that will be 40 feet wide. Tanya worked out the function $h = 0.02x^2 - 0.8x + 15$ to describe the arc of the bridge. In her rule, x gives horizontal distance from the entry point toward the other side of the river, and h gives height of the bridge above the water, both measured in feet.

- Sketch a graph of this function for $0 \leq x \leq 40$. On the sketch, show the coordinates of the starting and ending points and the point where the bridge surface is closest to the water.
- Show how coordinates for those key points on the graph can be calculated exactly from the rule, not simply estimated by tracing the graph or scanning a table.
- Write and estimate solutions for an equation that will locate point(s) on the bridge surface that are 10 feet above the water.

2 Mike's Water Slide Mike wanted to design a parabolic water slide for customers. The curve he wanted is shown in the following sketch. He wanted to have the entry to the slide at the point $(-10, 35)$, the low point to be $(0, 5)$, and the exit point to be $(5, 12.5)$.



- What rule will define a function with a graph in the shape that Mike wants?
- Use the rule from Part a to write and solve an equation that gives the x -coordinate(s) of point(s) on the slide that are exactly 20 feet above the ground.
- Mike tried to find the rule he wanted by looking first at functions like $y = ax^2$. He started by trying to find a so that $(-10, 30)$ and $(0, 0)$ would be points on the graph.
 - How do you suppose Mike was thinking about the problem that led him to this approach?
 - How could he have figured out the value of a with that start?
 - How could he then adjust the rule so the graph would go through the point $(-10, 35)$?
 - Verify that the graph of the adjusted rule in part iii goes through the point $(5, 12.5)$.

- 3 Flying Off the Slide** Mike figured that people would come off the end of his water slide at the point $(5, 12.5)$ with an upward speed of about 20 feet per second. He wanted to know about the resulting flight into the air and down to a splash in the pool lying at the end of the slide.

- What rule would give good estimates of the slider's height above the water as a function of time (in seconds) after they leave the end of the slide?
- What would be the maximum height of the slider?
- How long would it take the slider to make the trip from the end of the slide, into the air, and down to the water?



- 4 Water Slide Business** Tanya liked Mike's water slide idea, but she wondered how much they should charge customers for the experience. She asked a market research company to survey how the number of customers would depend on the admission price.
- The market research data suggested that daily number of customers n would be related to admission price x by $n = 250 - 10x$. What function shows how predicted income depends on admission price?
 - What price gives maximum daily income? Explain how you can locate that point by estimation and by reasoning that gives an exact answer.
 - If Mike and Tanya expect operation of the water slide attraction to cost \$450 per day, what function shows how predicted daily profit depends on admission price?
 - What water slide admission price leads to maximum profit?
 - What water slide admission price leads to a break-even operation of the water slide? Explain how you can locate the point(s) by estimation and by reasoning that gives an exact answer.
- 5** Use algebraic methods to find exact solutions for these quadratic equations. Show the steps in your reasoning and a check of each solution.
- $2x^2 + 11x + 12 = 0$
 - $6x^2 + 10x = 0$
 - $2x^2 + 11 = 19$

Summarize the Mathematics

When two variables are related by a quadratic function, that relationship can be recognized from patterns in tables and graphs of (x, y) data, from the rules that show how to calculate values of one variable from given values of the other, and from key features of the problem situations.

- a** Sketch two graphs illustrating the basic patterns that are modeled well by quadratic functions.
 - i. For each graph, write a brief explanation of the pattern shown in the graph and describe a problem situation that involves the pattern.
 - ii. Then give “ $y = \dots$ ” rules that would produce each given graph pattern and explain how those rules alone could be used to determine the pattern of change in the dependent variable as the independent variable increases.
- b** Suppose that you develop or discover a rule that shows how a variable y is a quadratic function of another variable x . Describe the different strategies you could use to complete tasks like these:
 - i. Find the value of y associated with a specific given value of x .
 - ii. Find the value of x that gives a specific target value of y .
 - iii. Describe the way that the value of y changes as the value of x increases or decreases.
- c** What information about the graph of a quadratic function can be obtained easily from each of these types of rules?
 - i. $y = ax^2 + c$
 - ii. $y = ax^2 + bx$
 - iii. $y = ax(x - m)$
- d** For questions that call for solving quadratic equations,
 - i. How many solutions would you expect, and how is that shown by the graphs of quadratic functions? By the quadratic formula?
 - ii. How would you decide on a solution strategy?
 - iii. How would you find the solution(s)?
 - iv. How would you check the solution(s)?

Be prepared to share your examples and descriptions with the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.

